MA121 Tutorial Problems #3

1. Let f be the function defined by

$$f(x) = \left\{ \begin{array}{cc} \frac{8x^3+4x-3}{2x-1} & \text{if } x \neq 1/2 \\ 5 & \text{if } x = 1/2 \end{array} \right\}.$$

Show that f is continuous at all points.

2. Evaluate each of the following limits:

$$\lim_{x\to+\infty} \frac{6x^2-5}{2-3x^2}, \qquad \lim_{x\to-\infty} \frac{6x^3-5x^2+2}{1-3x+x^4}.$$

3. Find the maximum value of $f(x) = 3x^4 - 16x^3 + 18x^2$ over the closed interval $[-1, 2]$.

4. Find the minimum value of $f(x) = (2x^2 - 5x + 2)^3$ over the closed interval $[0, 1]$.

5. Find the values of x for which $f'(x) = 0$ in each of the following cases:

$$f(x) = \frac{x^2}{1+x^2}, \qquad f(x) = x(x^2-9)^4.$$

6. Show that the polynomial $f(x) = x^3 - 3x + 1$ has three roots in the interval $(-2, 2)$. As a hint, you might wish to compute the values of f at the points ± 2, ± 1 and 0.

7. Show that the polynomial $f(x) = x^3 - 4x^2 - 3x + 1$ has exactly one root in $[0, 2]$.

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Some hints

1. Since f agrees with a rational function on the open interval $(-\infty, 1/2)$, it is continuous on that interval by a result of ours. Similarly, f is continuous on $(1/2, +\infty)$ as well, so it remains to check continuity at $y = 1/2$. In other words, you have to check that

$$\lim_{x \to 1/2} f(x) = f(1/2).$$

2. To compute the limit of a rational function as $x \to \pm\infty$, divide both the numerator and the denominator by the highest power of x that appears in the denominator.

3. Since you are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In this case,

$$f'(x) = 12x(x-1)(x-3).$$

Make sure that you only consider points which lie on the given closed interval.

4. Since you are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In this case,

$$f'(x) = 3(2x^2 - 5x + 2)^2(4x - 5).$$

Make sure that you only consider points which lie on the given closed interval.

5. In the former case, one has to use the quotient rule to eventually get

$$f'(x) = \frac{2x}{(1+x^2)^2}.$$

In the latter case, one has to use both the product and the chain rule to eventually get

$$f'(x) = 9(x^2-1)(x^2-9)^3.$$

6. Use Bolzano's theorem to find a root in $(-2, -1)$, a root in $(0, 1)$ and one in $(1, 2)$.

7. The fact that there exists a root in $[0, 2]$ follows by Bolzano's theorem. Suppose now that there exist two roots in $[0, 2]$. Then $f'(x)$ must have a root in $[0, 2]$ by Rolle's theorem. Find the roots of $f'(x)$ to see that this is not the case.