MA121 Tutorial Problems #2

- 1. Show that there exists some 0 < x < 1 such that $4x^3 + 3x = 2x^2 + 2$.
- 2. Evaluate each of the following limits:

$$\lim_{x \to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x + 1}, \qquad \lim_{x \to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x - 1}.$$

- **3.** Determine the values of x for which $6x^2 < 7x 2$.
- **4.** Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ccc} 2x & \text{if } x \in \mathbb{Q} \\ 5 - 3x & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Show that f is continuous at y = 1.

- 5. Suppose that f is continuous on [0, 1] and that 0 < f(x) < 1 for all $x \in [0, 1]$. Show that there exists some 0 < c < 1 such that f(c) = c.
- **6.** Determine the values of x for which $x^3 < 9x$.
- **7.** Let f be the function defined by

$$f(x) = \left\{ \begin{array}{cc} 3x & \text{if } x \le 1\\ 4x - 1 & \text{if } x > 1 \end{array} \right\}.$$

Show that f is continuous at all points.

8. Show that the function f defined by

$$f(x) = \left\{ \begin{array}{cc} 2x & \text{if } x \le 1\\ x+2 & \text{if } x > 1 \end{array} \right\}$$

is discontinuous at y = 1.

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Some hints

- 1. You need $f(x) = 4x^3 + 3x 2x^2 2$ to have a root in (0, 1). Since f is continuous on the closed interval [0, 1], you can apply Bolzano's theorem.
- 2. For the first limit, one can substitute x = 1 because rational functions are continuous wherever they are defined. For the second limit, you will need to simplify the fraction (using division of polynomials) before you can actually substitute.
- **3.** You need to find the values of x for which $6x^2 7x + 2 < 0$. Determine the roots of the quadratic and then factor.
- 4. You did something similar in your last homework. Let $\varepsilon > 0$ be given and note that

$$|f(x) - f(1)| = |f(x) - 2| = \left\{ \begin{array}{ll} 2|x - 1| & \text{if } x \in \mathbb{Q} \\ 3|1 - x| & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

You need to find some $\delta > 0$ such that $|x - 1| < \delta \implies |f(x) - f(1)| < \varepsilon$.

- 5. You need the function g(x) = f(x) x to have a root in (0, 1). Argue that this function is continuous on the closed interval [0, 1] and then use Bolzano's theorem.
- **6.** Move everything on the left hand side and factor: $x^3 < 9x \iff x(x-3)(x+3) < 0$.
- 7. In case you have your notes with you, Example 2.13 is quite similar. Continuity on the open interval $(-\infty, 1)$ follows by a lemma of ours, and so does continuity on $(1, \infty)$. It thus remains to check continuity at y = 1; use the ε - δ definition to do that.
- 8. Suppose f is continuous at y = 1. Then there exists some $\delta > 0$ such that

$$|x-1| < \delta \implies |f(x)-2| < 1.$$

In other words, there exists some $\delta > 0$ such that

$$1 - \delta < x < 1 + \delta \implies 1 < f(x) < 3.$$

Now, look at the point $x = 1 + \delta/2$ to see that this is a contradiction.

• The intuition behind the argument above: when $x \approx 1$, we have $2x \approx 2$ and $x + 2 \approx 3$, so the graph of f has a jump of length 1 near y = 1. Thus, the ε - δ definition of continuity is not expected to hold when $\varepsilon = 1$.