MA121 Tutorial Problems #1

1. Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.

(a) $A = \left\{n \in \mathbb{N} : \frac{1}{n} > \frac{1}{3}\right\}$

(b) $B = \{x \in \mathbb{R} : x > 1 \text{ and } 2x \leq 5\}$

(c) $C = \{x \in \mathbb{Z} : x > 1 \text{ and } 2x \leq 5\}$

(d) $D = \{x \in \mathbb{R} : x < y \text{ for all } y > 0\}$

(e) $E = \{x \in \mathbb{R} : x > y \text{ for all } y > 0\}$

2. Let $x \in \mathbb{R}$ be such that $x > -1$. Show that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

3. Let $f(x) = x^2 - 4x$ for all $x \in \mathbb{R}$. Show that $\inf f(x) = -4$, whereas $\inf\limits_{0 \leq x \leq 1} f(x) = -3$.

4. Let A, B be nonempty subsets of $\mathbb{R}$ such that $\sup A < \sup B$. Show that there exists an element $b \in B$ which is an upper bound of A.

5. Given any real number $x \neq 1$, show that

$$1 + x + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for all } n \in \mathbb{N}.$$

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Some hints

1a. This set contains only two elements.

1b. This set is actually an interval, namely $B = (1, 5/2]$.

1c. This set contains only one element.

1d. The elements of this set must be smaller than all positive reals. Draw the real line and ask yourselves: where are the positive reals? where can the elements of the set be? which are the elements of the set? why am I reading this hint?

1e. The elements of this set must be bigger than all positive reals. Is there any real number with this property?

2. Use induction: prove when $n = 1$, assume for some $n \in \mathbb{N}$ and then prove for $n + 1$.

3. For the first part, try to show that $f(x) + 4 \geq 0$ for all x and then argue that -4 is the minimum value attained by the function; why is it attained? Once a minimum is known to exist, an infimum also does and the two are equal.

For the second part, you need to show that $f(x) + 3 \geq 0$ for all $0 \leq x \leq 1$. The rest of the argument is very similar to the argument described above.

4. This problem is not as hard as it may seem. You know that $\sup A < \sup B$ and you want to somehow use this fact to get started. Note that a number "smaller than the least upper bound" cannot be an upper bound. In this case, $\sup A$ cannot be an upper bound of B, so it cannot be larger than all elements of B, hence some element $b \in B$ is bigger. This shows that some $b \in B$ is such that $b > \sup A$. The rest of the argument should be easy.

5. Use induction: prove when $n = 1$, assume for some $n \in \mathbb{N}$ and then prove for $n + 1$.