MA121 Christmas exam
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \left\{n \in \mathbb{N} : \frac{n}{2} \in \mathbb{N}\right\}$

(b) $B = \{x \in \mathbb{R} : 2x > 3\}$

(c) $C = \{x \in \mathbb{R} : x < y \textit{ for all } y > 0\}$

(d) $D = \{x \in \mathbb{R} : 4x^2 \leq 4x - 1\}$

• A complete list of answers is provided by the following table.

	min	inf	max	sup
A	2	2	DNE	DNE
B	DNE	3/2	DNE	DNE
C	DNE	DNE	0	0
D	1/2	1/2	1/2	1/2

• The set A contains all even natural numbers; this means that $A = \{2, 4, 6, \ldots\}$.

• The set B contains all real numbers x with $x > 3/2$; this means that $B = (3/2, +\infty)$.

• The set C contains the real numbers x which are smaller than all positive reals; this means that $C = (-\infty, 0]$.

• The set D contains all real numbers x with $4x^2 - 4x + 1 \leq 0$. Since this inequality can also be written as $(2x-1)^2 \leq 0$, however, it is easy to see that $D = \{1/2\}$.

2. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} \frac{4x^3-7x+3}{2x-1} & \textit{if } x \neq 1/2 \\ -2 & \textit{if } x = 1/2 \end{array} \right\}.$$

Show that f is continuous at $y = 1/2$. As a hint, one may avoid the ε-δ definition here.

To check continuity at $y = 1/2$, we have to check that

$$\lim_{x \to 1/2} f(x) = f(1/2).$$

Using division of polynomials to evaluate the limit, one now finds that

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} \frac{4x^3 - 7x + 3}{2x - 1} = \lim_{x \to 1/2} (2x^2 + x - 3).$$

Since limits of polynomials can be computed by simple substitution, this also implies

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} (2x^2 + x - 3) = 2 \cdot \frac{1}{4} + \frac{1}{2} - 3 = -2 = f(1/2).$$

3. *Show that the polynomial* $f(x) = x^4 - 2x^3 + x^2 - 1$ *has exactly one root in* $(1, 2)$.

Being a polynomial, f is continuous on the closed interval $[1, 2]$ and we also have

$$f(1) = -1 < 0, \qquad f(2) = 3 > 0.$$

Thus, f has a root in $(1, 2)$ by Bolzano's theorem. Suppose it has two roots in $(1, 2)$. In view of Rolle's theorem, f' must then have a root in $(1, 2)$ as well. On the other hand,

$$f'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1)$$

and the roots of this function are $x = 0$ as well as

$$x = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2 \cdot 2} = \frac{3 \pm 1}{4} \qquad \Longrightarrow \qquad x = 1, \quad x = 1/2.$$

Since none of those lies in $(1, 2)$, we conclude that f cannot have two roots in $(1, 2)$.

4. *Find the maximum value of* $f(x) = (2x - 5)^2(5 - x)^3$ *over the closed interval* $[2, 5]$.

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$f'(x) = 2(2x - 5)(2x - 5)' \cdot (5 - x)^3 + (2x - 5)^2 \cdot 3(5 - x)^2(5 - x)'$$

by the product and the chain rule. We simplify this expression and factor to get

$$\begin{aligned} f'(x) &= 4(2x - 5) \cdot (5 - x)^3 - 3(2x - 5)^2 \cdot (5 - x)^2 \\ &= (2x - 5)(5 - x)^2 \cdot (20 - 4x - 6x + 15) \\ &= (2x - 5)(5 - x)^2 \cdot 5(7 - 2x). \end{aligned}$$

Keeping this in mind, the only points at which the maximum value may occur are

$$x = 5/2, \qquad x = 5, \qquad x = 7/2, \qquad x = 2.$$

Note that each of these points lies in the closed interval $[2, 5]$ and that

$$f(5/2) = f(5) = 0, \qquad f(7/2) = 27/2, \qquad f(2) = 27.$$

Based on these facts, we may finally conclude that the maximum value is $f(2) = 27$.

5. *Let* f *be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 2 - 2x & \text{if } x < 1 \\ 4 - 5x & \text{if } x \geq 1 \end{array} \right\}.$$

Show that f *is discontinuous at* $y = 1$.

We will show that the ε-δ definition of continuity fails when $\varepsilon = 1$. Suppose it does not fail. Since $f(1) = -1$, there must then exist some $\delta > 0$ such that

$$|x - 1| < \delta \qquad \Longrightarrow \qquad |f(x) + 1| < 1. \tag{$*$}$$

Let us now examine the last equation for the choice $x = 1 - \frac{\delta}{2}$. On one hand, we have

$$|x - 1| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x) + 1| = |2 - 2x + 1| = 3 - 2x = 1 + \delta > 1$$

because $x = 1 - \frac{\delta}{2} < 1$ here. This actually violates the conclusion in equation $(*)$.

6. *Let $x \in \mathbb{R}$ be a real number such that $2 - nx \geq 0$ for all $n \in \mathbb{N}$. Show that $x \leq 0$.*

Suppose that $x > 0$, instead. Then it must be the case that

$$2 - nx \geq 0 \quad \Longrightarrow \quad nx \leq 2 \quad \Longrightarrow \quad n \leq \frac{2}{x} \qquad \text{for all } n \in \mathbb{N}.$$

This makes $2/x$ an upper bound of $\mathbb{N}$, violating the fact that $\mathbb{N}$ has no upper bound.

7. *Show that $3x^4 + 4x^3 \geq 12x^2 - 32$ for all $x \in \mathbb{R}$.*

We need to show that $f(x) = 3x^4 + 4x^3 - 12x^2 + 32$ is non-negative for all values of x. Let us then try to compute the minimum value of this function. We have

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x - 1)(x + 2)$$

and so the sign of f' can be determined using the table below.

x		-2	0	1
$12x$	$-$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

According to this table, the minimum value is either $f(-2) = 0$ or else $f(1) = 27$. Since the former is smaller and also attained, we deduce that $\min f(x) = 0$, as needed.

8. *Show that the set $A = \{\frac{n+1}{n} : n \in \mathbb{N}\}$ is such that* $\inf A = 1$.

To see that 1 is a lower bound of the given set, we note that

$$n \in \mathbb{N} \quad \Longrightarrow \quad n + 1 > n \quad \Longrightarrow \quad \frac{n+1}{n} > 1.$$

To see that 1 is the greatest lower bound, suppose that $x > 1$ and note that

$$\frac{n+1}{n} < x \quad \Longleftrightarrow \quad n + 1 < nx \quad \Longleftrightarrow \quad 1 < n(x - 1) \quad \Longleftrightarrow \quad \frac{1}{x - 1} < n.$$

According to one of our theorems, we can always find an integer $n \in \mathbb{N}$ such that $n > \frac{1}{x-1}$. Then our computation above shows that $\frac{n+1}{n} < x$. In particular, x is strictly larger than an element of A, so x cannot possibly be a lower bound of A.