MA121, Sample Exam #2
Solutions

1. *Compute each of the following integrals:*

$$\int \frac{3x+5}{x^3-x}\,dx, \qquad \int x\cos x\,dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{3x+5}{x^3-x} = \frac{3x+5}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \qquad (*)$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$3x+5 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

and we can now look at some suitable choices of x to find

$$x=0, \quad x=-1, \quad x=1 \quad \Longrightarrow \quad 5=-A, \quad 2=2B, \quad 8=2C.$$

This means that $A=-5$, $B=1$ and $C=4$. In particular, equation $(*)$ reduces to

$$\frac{3x+5}{x^3-x} = -\frac{5}{x} + \frac{1}{x+1} + \frac{4}{x-1}$$

and we may integrate this equation term by term to conclude that

$$\int \frac{3x+5}{x^3-x}\,dx = -5\log|x| + \log|x+1| + 4\log|x-1| + C.$$

- To compute the second integral, we integrate by parts to find that

$$\int x\cos x\,dx = \int x(\sin x)'\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

2. *Let f be defined by $f(x) = \left\{\begin{array}{ll} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array}\right\}$. Show that f is integrable on $[0,1]$.*

- Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$, we must clearly have

$$\inf_{[x_0,x_1]} f(x) = 0, \qquad \inf_{[x_k,x_{k+1}]} f(x) = 1 \quad \text{for each } 1 \leq k \leq n-1$$

because $[x_0, x_1]$ is the only subinterval which contains the point $x=0$. This gives

$$\begin{aligned} S^-(f,P) &= \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1}-x_k) \\ &= 0(x_1-x_0) + (x_2-x_1) + (x_3-x_2) + \ldots + (x_n - x_{n-1}) \\ &= x_n - x_1 = 1 - x_1. \end{aligned}$$

Taking the supremum over all possible partitions, we then find that

$$\sup_P \{S^-(f,P)\} = \sup_{0<x_1<1} (1-x_1) = 1.$$

To similarly compute the Darboux upper sums of f, let us start by noting that

$$\sup_{[x_k,x_{k+1}]} f(x) = 1 \quad \text{for each } 0 \leq k \leq n-1$$

because every subinterval $[x_k, x_{k+1}]$ contains a nonzero number. This gives

$$\begin{aligned} S^+(f,P) &= \sum_{k=0}^{n-1} \sup_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1}-x_k) \\ &= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) \\ &= x_n - x_0 = 1 - 0 = 1. \end{aligned}$$

Taking the infimum over all possible partitions, we may then finally conclude that

$$\inf_P \{S^+(f,P)\} = \inf_P \{1\} = 1 = \sup_P \{S^-(f,P)\}.$$

3. *Define a sequence* $\{a_n\}$ *by setting* $a_1 = 4$ *and*

$$a_{n+1} = \frac{1}{5-a_n} \quad \textit{for each } n \geq 1.$$

Show that $0 \leq a_{n+1} \leq a_n \leq 4$ *for each* $n \geq 1$, *use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 4$ and $a_2 = 1/(5-a_1) = 1$, the statement

$$0 \leq a_{n+1} \leq a_n \leq 4$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 0 \geq -a_{n+1} \geq -a_n \geq -4 \quad &\Longrightarrow \quad 5 \geq 5 - a_{n+1} \geq 5 - a_n \geq 1 \\ &\Longrightarrow \quad 1/5 \leq a_{n+2} \leq a_{n+1} \leq 1 \\ &\Longrightarrow \quad 0 \leq a_{n+2} \leq a_{n+1} \leq 4. \end{aligned}$$

Thus, the statement holds for $n+1$ as well, so it must actually hold for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \frac{1}{5-a_n} \quad \Longrightarrow \quad L = \frac{1}{5-L} \quad \Longrightarrow \quad L^2 - 5L + 1 = 0.$$

Solving this quadratic equation now gives

$$L = \frac{5 \pm \sqrt{25-4}}{2} = \frac{5 \pm \sqrt{21}}{2}\,.$$

Since $0 \leq a_n \leq 4$ for each $n \in \mathbb{N}$, however, we must also have $0 \leq L \leq 4$, so

$$L = \frac{5 - \sqrt{21}}{2}\,.$$

4. *Compute each of the following limits:*

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}\,, \qquad \lim_{x\to\infty} x\sin(1/x).$$

- Since the first limit is a $0/0$ limit, we may apply L'Hôpital's rule to find that

$$L = \lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}\,.$$

Since this is still a $0/0$ limit, L'Hôpital's rule is still applicable and we get

$$L = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}\,.$$

- When it comes to the second limit, we can express it in the form

$$M = \lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}\,.$$

This is now a $0/0$ limit, so L'Hôpital's rule becomes applicable and we get

$$M = \lim_{x\to\infty} \frac{\cos(1/x)\cdot(1/x)'}{(1/x)'} = \lim_{x\to\infty} \cos(1/x) = \cos 0 = 1.$$

5. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,n^2}{1+n^4}\,, \qquad \sum_{n=1}^{\infty} \left(\frac{2n}{1+3n}\right)^n.$$

- To test the first series for convergence, we use the alternating series test with

$$a_n = \frac{n^2}{1+n^4}\,.$$

Note that a_n is certainly non-negative for each $n \geq 1$, and that we also have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{1+n^4} = \lim_{n\to\infty} \frac{1/n^2}{1/n^4+1} = \frac{0}{0+1} = 0.$$

Moreover, a_n is decreasing for each $n \geq 1$ because

$$\left(\frac{n^2}{1+n^4}\right)' = \frac{2n(1+n^4) - 4n^3\cdot n^2}{(1+n^4)^2} = \frac{2n(1-n^4)}{(1+n^4)^2} \leq 0$$

whenever $n \geq 1$. Thus, the given series converges by the alternating series test.

- To test the second series for convergence, we use the comparison test. Since

$$\left(\frac{2n}{1+3n}\right)^n \leq \left(\frac{2n}{3n}\right)^n = \left(\frac{2}{3}\right)^n,$$

the given series is smaller than a convergent geometric series, so it converges itself.

6. *Find the radius of convergence of the power series*

$$f(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

- To find the radius of convergence, one always uses the ratio test. In our case,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = \frac{(n+1)^2 \cdot x}{(2n+1)(2n+2)}$$

and this implies that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n^2+2n+1)\,|x|}{4n^2+6n+2} = \frac{|x|}{4}.$$

Thus, the power series converges when $|x|/4 < 1$ and diverges when $|x|/4 > 1$. In other words, it converges when $|x| < 4$ and diverges when $|x| > 4$. This also means that $R = 4$.

7. *Suppose f is a differentiable function such that $f'(x) = 2x \cdot f(x)$ for all $x \in \mathbb{R}$. Show that there exists some constant C such that $f(x) = Ce^{x^2}$ for all $x \in \mathbb{R}$.*

- Letting $g(x) = f(x) \cdot e^{-x^2}$ for convenience, one easily finds that

$$\begin{aligned} g'(x) &= f'(x) \cdot e^{-x^2} + f(x) \cdot e^{-x^2} \cdot (-2x) \\ &= e^{-x^2} \cdot [f'(x) - 2x \cdot f(x)] = 0. \end{aligned}$$

In particular, $g(x)$ is actually constant, say $g(x) = C$ for all $x \in \mathbb{R}$, and this implies

$$g(x) = C \quad\Longrightarrow\quad f(x) \cdot e^{-x^2} = C \quad\Longrightarrow\quad f(x) = Ce^{x^2}.$$

8. *Suppose that f is a function with*

$$|f(x) - f(y)| \leq |x-y|^2 \quad \textit{for all } x, y \in \mathbb{R}.$$

Using the limit definition of the derivative, show that f is actually constant.

- We need only show that $f'(y) = 0$ for all $y \in \mathbb{R}$. Using the given inequality, we get

$$0 \leq \frac{|f(x) - f(y)|}{|x-y|} \leq |x-y| \quad \text{whenever } x \neq y.$$

Since $|x-y|$ approaches zero as $x \to y$, the quotient above is thus squeezed between two functions which approach zero as $x \to y$. Using the Squeeze Law, we conclude that the quotient itself must approach zero as $x \to y$. This also implies that $f'(y) = 0$.