MA121, Sample Exam #1
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \{n \in \mathbb{N} : n - 1 \in \mathbb{N}\}$

(b) $B = \{x \in \mathbb{R} : 2x \leq 5\}$

(c) $C = \{x \in \mathbb{R} : |x| < y \text{ for all } y > 0\}$

(d) $D = \{x \in \mathbb{R} : |x + 1| < 1\}$

• A complete list of answers is provided by the following table.

	min	inf	max	sup
A	2	2	DNE	DNE
B	DNE	DNE	5/2	5/2
C	0	0	0	0
D	DNE	-2	DNE	0

• The set A contains all $n \in \mathbb{N}$ with $n - 1 \geq 1$; this means that $A = \{2, 3, 4, \ldots\}$.

• The set B contains all real numbers x with $x \leq 5/2$; this means that $B = (-\infty, 5/2]$.

• The set C contains all real numbers x with $-y < x < y$ for all $y > 0$. In particular, x must be bigger than all negative reals and smaller than all positive reals, so $C = \{0\}$.

• The set D contains all real numbers x whose distance from -1 is strictly less than 1. Based on this fact, it is easy to see that $D = (-2, 0)$.

2. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cl} \frac{4x^3-7x-3}{2x-3} & \text{if } x \neq 3/2 \\ 10 & \text{if } x = 3/2 \end{array} \right\}.$$

Show that f is continuous at $y = 3/2$. As a hint, one may avoid the ε-δ definition here.

To check continuity at $y = 3/2$, we have to check that

$$\lim_{x \to 3/2} f(x) = f(3/2).$$

Using division of polynomials to evaluate the limit, one now finds that

$$\lim_{x \to 3/2} f(x) = \lim_{x \to 3/2} \frac{4x^3 - 7x - 3}{2x - 3} = \lim_{x \to 3/2} (2x^2 + 3x + 1).$$

Since limits of polynomials can be computed by simple substitution, this also implies

$$\lim_{x \to 3/2} f(x) = \lim_{x \to 3/2} (2x^2 + 3x + 1) = 2 \cdot \frac{9}{4} + 3 \cdot \frac{3}{2} + 1 = 10 = f(3/2).$$

3. *Show that there exists some* $0 < x < 1$ *such that* $(x^2 - 2x + 3)^3 = (2x^2 - x + 1)^4$.

Let $f(x) = (x^2 - 2x + 3)^3 - (2x^2 - x + 1)^4$ for all $x \in [0, 1]$. Being a polynomial, f is then continuous on the closed interval $[0, 1]$. Once we now note that

$$f(0) = 3^3 - 1^4 = 26 > 0, \qquad f(1) = 2^3 - 2^4 = -8 < 0,$$

we may use Bolzano's theorem to conclude that $f(x) = 0$ for some $x \in (0, 1)$. This also implies that $(x^2 - 2x + 3)^3 = (2x^2 - x + 1)^4$ for some $0 < x < 1$, as needed.

4. *Find the maximum value of* $f(x) = x(7 - x^2)^3$ *over the closed interval* $[-1, 3]$.

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$f'(x) = 1 \cdot (7 - x^2)^3 + x \cdot 3(7 - x^2)^2 \cdot (7 - x^2)'$$

by the product and the chain rule. We simplify this expression and factor to get

$$\begin{aligned} f'(x) &= (7 - x^2)^3 + 3x(7 - x^2)^2 \cdot (-2x) = (7 - x^2)^2 \cdot (7 - x^2 - 6x^2) \\ &= (7 - x^2)^2 \cdot 7(1 - x^2). \end{aligned}$$

Keeping this in mind, the only points at which the maximum value may occur are

$$x = 3, \qquad x = \pm 1, \qquad x = \pm\sqrt{7}.$$

We exclude the point $x = -\sqrt{7}$, as this fails to lie in $[-1, 3]$, and we now compute

$$f(3) = -24, \qquad f(\pm 1) = \pm 216, \qquad f(\sqrt{7}\,) = 0.$$

Based on these facts, we may finally conclude that the maximum value is $f(1) = 216$.

5. *Suppose that* f *is a differentiable function such that*

$$f'(x) = \frac{1}{1 + x^2} \quad \textit{for all } x \in \mathbb{R}.$$

Show that $f(x) + f(1/x) = 2f(1)$ *for all* $x > 0$.

Let us set $g(x) = f(x) + f(1/x)$ for convenience. Using the chain rule, we then get

$$\begin{aligned} g'(x) &= f'(x) + f'(1/x) \cdot (1/x)' \\ &= \frac{1}{1 + x^2} + \frac{1}{1 + 1/x^2} \cdot \left(-\frac{1}{x^2} \right) \\ &= \frac{1}{1 + x^2} - \frac{1}{x^2 + 1} = 0. \end{aligned}$$

This shows that $g(x)$ is constant, and it also implies that $g(x) = g(1) = 2f(1)$.

6. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 2-3x & \text{if } x \leq 2 \\ 4-5x & \text{if } x > 2 \end{array} \right\}.$$

Show that f is discontinuous at $y = 2$.

We will show that the ε-δ definition of continuity fails when $\varepsilon = 2$. Suppose it does not fail. Since $f(2) = -4$, there must then exist some $\delta > 0$ such that

$$|x-2| < \delta \quad \Longrightarrow \quad |f(x)+4| < 2. \tag{$*$}$$

Let us now examine the last equation for the choice $x = 2 + \frac{\delta}{2}$. On one hand, we have

$$|x-2| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x)+4| = |4-5x+4| = 5x-8 = 2 + \frac{5\delta}{2} > 2$$

because $x = 2 + \frac{\delta}{2} > 2$ here. This actually violates the conclusion in equation $(*)$.

7. *Let A be a nonempty subset of $\mathbb{R}$ that has an upper bound and let $\varepsilon > 0$ be given. Show that there exists some element $a \in A$ such that* $\sup A - \varepsilon < a \leq \sup A$.

Note that $\sup A - \varepsilon$ cannot be an upper bound of A because it is smaller than the least upper bound of A. This means that some $a \in A$ is such that $a > \sup A - \varepsilon$. On the other hand, we must also have $a \leq \sup A$ because $a \in A$ and $\sup A$ is an upper bound of A.

8. *Show that the polynomial $f(x) = x^4 - 2x^3 + x^2 - 1$ has exactly one root in $(1,2)$.*

Being a polynomial, f is continuous on the closed interval $[1,2]$ and we also have

$$f(1) = -1 < 0, \qquad f(2) = 3 > 0.$$

Thus, f has a root in $(1,2)$ by Bolzano's theorem. Suppose it has two roots in $(1,2)$. In view of Rolle's theorem, f' must then have a root in $(1,2)$ as well. On the other hand,

$$f'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1)$$

and the roots of this function are $x = 0$ as well as

$$x = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2 \cdot 2} = \frac{3 \pm 1}{4} \quad \Longrightarrow \quad x = 1, \quad x = 1/2.$$

Since none of those lies in $(1,2)$, we conclude that f cannot have two roots in $(1,2)$.