MA121, Homework #8
Solutions

1. *Letting $f(x,y)=\log(x^2+y^2)$, find the rate at which f is changing at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 6,8\rangle$.*

- To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, we need to divide $\mathbf{v}$ by its length, namely

$$||\mathbf{v}||=\sqrt{6^2+8^2}=10 \quad\Longrightarrow\quad \mathbf{u}=\frac{1}{10}\,\mathbf{v}=\langle 6/10, 8/10\rangle\,.$$

The desired rate of change is given by the directional derivative $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}$. Since

$$\nabla f(x,y)=\langle f_x,f_y\rangle=\left\langle \frac{2x}{x^2+y^2}\,,\frac{2y}{x^2+y^2}\right\rangle \quad\Longrightarrow\quad \nabla f(1,2)=\langle 2/5,4/5\rangle\,,$$

we may thus conclude that the desired rate of change is

$$D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=\frac{2}{5}\cdot\frac{6}{10}+\frac{4}{5}\cdot\frac{8}{10}=\frac{44}{50}=\frac{22}{25}\,.$$

2. *Assuming that $z=f(x^2-y^2,y^2-x^2)$ for some differentiable function f, show that*

$$yz_x+xz_y=0.$$

- In this case, $z=f(u,v)$ with $u=x^2-y^2$ and $v=y^2-x^2$, so the chain rule gives

$$\begin{aligned} z_x &= f_x = f_u u_x + f_v v_x = 2xf_u - 2xf_v,\\ z_y &= f_y = f_u u_y + f_v v_y = -2yf_u + 2yf_v. \end{aligned}$$

Once we now combine these two equations, we get the desired identity

$$yz_x+xz_y=2xyf_u-2xyf_v-2xyf_u+2xyf_v=0.$$

3. *Find the maximum value of $f(x,y)=x^2+xy+3x+2y$ over the region*

$$R=\{(x,y)\in\mathbb{R}^2 : x^2\le y\le 4\}.$$

- To find the critical points, we need to solve the equations

$$0=f_x(x,y)=2x+y+3, \qquad 0=f_y(x,y)=x+2.$$

In this case, $x=-2$ by the second equation, so $y=-2x-3=1$ by the first equation. In particular, $(-2,1)$ is the only critical point of f, however this point does not lie in the given region, so we may simply ignore it.

- Next, we check the points on the boundary. Along the parabola $y = x^2$, we have
$$f(x, y) = x^2 + x^3 + 3x + 2x^2 = x^3 + 3x^2 + 3x$$
and we need to find the maximum value of this function on $[-2, 2]$. Noting that
$$g(x) = x^3 + 3x^2 + 3x \quad \Longrightarrow \quad g'(x) = 3(x^2 + 2x + 1) = 3(x + 1)^2,$$
we see that the maximum value may only occur at $x = -2$, $x = 2$ or $x = -1$. Since
$$g(-2) = -2, \qquad g(2) = 26, \qquad g(-1) = -1,$$
the maximum value is $g(2) = 26$ and this corresponds to the value $f(2, 4) = 26$.
- It now remains to check the boundary points along the line $y = 4$. For these points,
$$f(x, y) = x^2 + 4x + 3x + 8 = x^2 + 7x + 8$$
and we need to find the maximum value of this function on $[-2, 2]$. Noting that
$$h(x) = x^2 + 7x + 8 \quad \Longrightarrow \quad h'(x) = 2x + 7 = 2(x + 7/2),$$
we see that the maximum value may only occur at $x = -2$ or $x = 2$. Since
$$h(-2) = 4 - 14 + 8 = -2, \qquad h(2) = 4 + 14 + 8 = 26,$$
the maximum value over the whole region is the value $f(2, 4) = 26$ we obtained above.

4. *Classify the critical points of $f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2$.*

- To find the critical points, we need to solve the equations
$$\begin{aligned} 0 &= f_x(x, y) = 4x^3 + 8xy^2 - 4x = 4x(x^2 + 2y^2 - 1), \\ 0 &= f_y(x, y) = 8x^2y + 4y = 4y(2x^2 + 1). \end{aligned}$$
In this case, the second equation gives $y = 0$ and then the first equation becomes
$$0 = 4x(x^2 - 1) = 4x(x + 1)(x - 1) \quad \Longrightarrow \quad x = -1, 0, 1.$$
Thus, there are three critical points and those are $(-1, 0)$, $(0, 0)$ and $(1, 0)$.
- To classify the critical points, we compute the Hessian matrix
$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x^2 + 8y^2 - 4 & 16xy \\ 16xy & 8x^2 + 4 \end{bmatrix}.$$
When it comes to the critical point $(0, 0)$, this gives
$$H = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Longrightarrow \quad \det H = -16 < 0$$
so the origin is a saddle point. When it comes to the critical points $(\pm 1, 0)$, we have
$$H = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix} \quad \Longrightarrow \quad \det H = 8 \cdot 12 > 0$$
and also $f_{xx} = 8 > 0$, so each of these points is a local minimum.

5. *Classify the critical points of* $f(x, y) = 3x^2 - 3y^2 + 8xy + 10x - 20y + 30$.

- To find the critical points, we need to solve the equations

$$\begin{aligned} 0 &= f_x(x, y) = 6x + 8y + 10, \\ 0 &= f_y(x, y) = 8x - 6y - 20. \end{aligned}$$

  We multiply the first equation by 6, the second equation by 8 and then we add to get

$$0 = 36x + 60 + 64x - 160 = 100x - 100 \quad \Longrightarrow \quad x = 1.$$

  The first equation now gives $8y = -16$, so the only critical point is the point $(1, -2)$.

- In order to classify this critical point, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}.$$

  Since $\det H = -36 - 64$ is negative, the critical point is actually a saddle point.