MA121, Homework #7
Solutions

1. *Compute each of the following sums in terms of known functions:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, 4^n x^{2n+1}}{(2n+1)!}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{9^n \cdot (2n)!}.$$

- Relating the first sum to the Taylor series for the exponential function, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!} = x\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!} = x\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = xe^{-x^2}.$$

- The second sum is related to the Taylor series for the sine function, namely

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, 4^n x^{2n+1}}{(2n+1)!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1} x^{2n+1}}{(2n+1)!} = \frac{\sin(2x)}{2}.$$

- Finally, the third sum is related to the Taylor series for the cosine function since

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{9^n \cdot (2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n\, (x/3)^{2n}}{(2n)!} = \cos(x/3) - 1.$$

2. *Show that the equation $r = 3\sin\theta$ describes a circle in polar coordinates. As a hint, you may wish to multiply this equation by r and then express it in terms of x and y.*

- Following the hint, we multiply by r and switch to rectangular coordinates; this gives

$$r = 3\sin\theta \quad\Longrightarrow\quad r^2 = 3r\sin\theta \quad\Longrightarrow\quad x^2 + y^2 = 3y.$$

We now move all terms to the left hand side and we complete the square to get

$$x^2 + y^2 - 3y = 0 \quad\Longrightarrow\quad x^2 + (y - 3/2)^2 = (3/2)^2.$$

This equation describes the points (x, y) whose distance from $(0, 3/2)$ is equal to $3/2$. In particular, it describes the circle of radius $3/2$ around $(0, 3/2)$.

3. *Compute the area of a right triangle whose sides have length a, b and $\sqrt{a^2 + b^2}$.*

- Suppose that the triangle has the points $(0, 0)$, $(a, 0)$ and (a, b) as its vertices. Then we can view it as the region that lies between the graph of $f(x) = bx/a$ and the x-axis, so its area is given by the formula

$$\text{Area} = \int_0^a f(x)\, dx = \int_0^a \frac{bx}{a}\, dx = \left[\frac{bx^2}{2a}\right]_0^a = \frac{a^2 b}{2a} = \frac{ab}{2}.$$

4. *Let R be the region between the graph of $f(x) = e^x - 1$ and the x-axis on $[0,1]$. Find the volume of the solid obtained upon rotation of R around the x-axis.*

- The desired volume is given by the formula

$$\begin{aligned}\text{Volume} &= \pi \int_0^1 f(x)^2\,dx = \pi \int_0^1 (e^x - 1)^2\,dx = \pi \int_0^1 (e^{2x} - 2e^x + 1)\,dx \\ &= \pi \left[\frac{e^{2x}}{2} - 2e^x + x \right]_0^1 = \frac{\pi(e^2 - 4e + 5)}{2}\,.\end{aligned}$$

5. *Compute each of the following limits:*

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2}\,, \qquad \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}\,, \qquad \lim_{(x,y)\to(1,1)} \frac{3x^2 - xy - 2y^2}{x-y}\,.$$

- Since rational functions are known to be continuous, it is clear that

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2} = \frac{1\cdot 2}{1^2+2^2} = \frac{2}{5}\,.$$

- To compute the second limit, we switch to polar coordinates and we write

$$f(x,y) = \frac{x^2y}{x^2+y^2} = \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2} = r\cos^2\theta\sin\theta.$$

  Since (x,y) approaches the origin, we have $r = \sqrt{x^2+y^2} \longrightarrow 0$ and so the given function must approach zero as well. More precisely, we have

$$0 \leq |f(x,y)| = |r\cos^2\theta\sin\theta| \leq r$$

  and the fact that $r \longrightarrow 0$ implies that $f(x,y) \longrightarrow 0$ because of the Squeeze Law.

- Using division of polynomials to compute the last limit, one finds that

$$\lim_{(x,y)\to(1,1)} \frac{3x^2 - xy - 2y^2}{x-y} = \lim_{(x,y)\to(1,1)} (3x+2y) = 3 + 2 = 5$$

  because the linear function $g(x,y) = 3x + 2y$ is known to be continuous.