MA121, Homework #6
Solutions

1. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, e^{1/n}}{n}.$$

- When it comes to the first series, we use the comparison test. Since $n \geq 1$, we have

$$1/n \leq 1 \quad \Longrightarrow \quad e^{1/n} \leq e \quad \Longrightarrow \quad \frac{e^{1/n}}{n^2} \leq \frac{e}{n^2}.$$

  Being smaller than a convergent p-series, the given series must thus be convergent itself.

- To test the second series for convergence, we use the limit comparison test with

$$a_n = \frac{e^{1/n}}{n}, \qquad b_n = \frac{1}{n}.$$

  Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} e^{1/n} = e^0 = 1.$$

  Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

- To test the last series for convergence, we use the alternating series test with

$$a_n = \frac{e^{1/n}}{n}.$$

  Note that a_n is certainly non-negative for each $n \geq 1$, and that we also have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = \lim_{n\to\infty} \frac{e^0}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

  Moreover, a_n is decreasing for each $n \geq 1$ because the derivative

$$\left(\frac{e^{1/n}}{n}\right)' = \frac{e^{1/n} \cdot (-n^{-2}) \cdot n - e^{1/n}}{n^2} = -\frac{e^{1/n}}{n^2} \cdot (n^{-1} + 1)$$

  is negative for each $n \geq 1$. Thus, the given series converges by the alternating series test.

2. *Find the radius of convergence for each of the following power series:*

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}.$$

- One always determines the radius of convergence using the ratio test. In this case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \frac{|x|}{3}$$

so the series converges when $|x|/3 < 1$ and diverges when $|x|/3 > 1$. In other words, the series converges when $|x| < 3$ and diverges when $|x| > 3$. This also means that $R = 3$.

- To find the radius of convergence for the second power series, we similarly compute

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} = |x| \cdot \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}} = |x|.$$

Since the series converges when $|x| < 1$ and diverges when $|x| > 1$, this yields $R = 1$.

3. *Differentiate the formula for a geometric series to show that*

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad \textit{whenever } |x| < 1.$$

- Since $|x| < 1$ by assumption, the formula for a geometric series is applicable, so we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}.$$

Differentiating both sides of this equation and multiplying by x, we now find that

$$\sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2} \quad \Longrightarrow \quad \sum_{n=0}^{\infty} nx^n = x(1-x)^{-2} = \frac{x}{(1-x)^2}.$$

4. *Use the nth term test to show that each of the following series diverges:*

$$\sum_{n=1}^{\infty} n^{1/n}, \qquad \sum_{n=1}^{\infty} n\sin(1/n).$$

- In each case, we have to show that the nth term fails to approach zero as $n \to \infty$. When it comes to the first series, this means that we have to compute the limit

$$L = \lim_{n\to\infty} n^{1/n} \quad \Longrightarrow \quad \log L = \lim_{n\to\infty} \log n^{1/n} = \lim_{n\to\infty} \frac{\log n}{n}.$$

Since the rightmost limit is an ∞/∞ limit, we may use L'Hôpital's rule to get

$$\log L = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \Longrightarrow \quad L = e^0 = 1.$$

- When it comes to the second series, we have to similarly compute the limit

$$L = \lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}.$$

Since this is a 0/0 limit, L'Hôpital's rule is still applicable and we find

$$L = \lim_{n\to\infty} \frac{\cos(1/n) \cdot (1/n)'}{(1/n)'} = \lim_{n\to\infty} \cos(1/n) = \cos 0 = 1.$$