MA121, Homework #5
Solutions

1. *Compute each of the following integrals:*

$$\int 2x^3 e^{x^2}\,dx, \qquad \int \frac{x^2+3}{x(x+1)^2}\,dx.$$

- For the first integral, we use the substitution $u = x^2$. This gives $du = 2x\,dx$, hence

$$\int 2x^3 e^{x^2}\,dx = \int 2x \cdot x^2 e^{x^2}\,dx = \int ue^u\,du.$$

Focusing on the rightmost integral, we integrate by parts to find that

$$\int ue^u\,du = \int u\,(e^u)'\,du = ue^u - \int e^u\,du = ue^u - e^u + C.$$

Once we now combine the last two equations, we get

$$\int 2x^3 e^{x^2}\,dx = \int ue^u\,du = ue^u - e^u + C = x^2 e^{x^2} - e^{x^2} + C.$$

- To compute the second integral, we use partial fractions to write

$$\frac{x^2+3}{x(x+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x+1)^2} \qquad (*)$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2 + 3 = A(x+1)^2 + (Bx+C)x$$

and we can now look at some suitable choices of x to find

$$\begin{aligned} x = 0 &\implies 3 = A \\ x = -1 &\implies 4 = B - C \\ x = 1 &\implies 4 = 4A + B + C = 12 + B + C. \end{aligned}$$

Adding the last two equations, we get $8 = 12 + 2B$, and this implies

$$2B = 8 - 12 = -4 \implies B = -2 \implies C = B - 4 = -6.$$

Once we now return to equation $(*)$, we may conclude that

$$\int \frac{x^2+3}{x(x+1)^2}\,dx = \int \frac{3}{x}\,dx - \int \frac{2x+6}{(x+1)^2}\,dx.$$

To compute the rightmost integral, we set $u = x + 1$. This gives $du = dx$, hence

$$\int \frac{2x+6}{(x+1)^2}\,dx = \int \frac{2(u-1)+6}{u^2}\,du = \int \frac{2}{u}\,du + \int \frac{4}{u^2}\,du$$
$$= 2\log|u| - 4u^{-1} + C = 2\log|x+1| - 4(x+1)^{-1} + C.$$

Combining the last two equations, we thus arrive at

$$\int \frac{x^2+3}{x(x+1)^2}\,dx = 3\log|x| - 2\log|x+1| + 4(x+1)^{-1} + C.$$

2. *Compute each of the following integrals:*

$$\int (\log x)^2\,dx, \qquad \int \frac{x^2+x-3}{x^2+3x+2}\,dx.$$

- To compute the first integral, we integrate by parts to find that

$$\int (\log x)^2\,dx = \int x'(\log x)^2\,dx = x(\log x)^2 - 2\int \log x\,dx.$$

Focusing on the rightmost integral, we integrate by parts once again to get

$$\int \log x\,dx = \int x'\log x\,dx = x\log x - \int x\cdot x^{-1}\,dx = x\log x - x + C.$$

Once we now combine the last two equations, we may conclude that

$$\int (\log x)^2\,dx = x(\log x)^2 - 2x\log x + 2x + C.$$

- To compute the second integral, we use division of polynomials to write

$$\frac{x^2+x-3}{x^2+3x+2} = 1 - \frac{2x+5}{x^2+3x+2} = 1 - \frac{2x+5}{(x+1)(x+2)}.$$

Note that we can integrate the rightmost expression using partial fractions, namely

$$\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$2x + 5 = A(x+2) + B(x+1)$$

and we can look at some suitable choices of x to find that

$$x = -1, \quad x = -2 \qquad \Longrightarrow \qquad 3 = A, \quad 1 = -B.$$

Combining our computations above, we thus arrive at the equation

$$\frac{x^2+x-3}{x^2+3x+2} = 1 - \frac{2x+5}{(x+1)(x+2)} = 1 - \frac{3}{x+1} + \frac{1}{x+2}.$$

Once we now integrate this equation term by term, we may finally conclude that

$$\int \frac{x^2+x-3}{x^2+3x+2}\,dx = x - 3\log|x+1| + \log|x+2| + C.$$

3. *Define a sequence* $\{a_n\}$ *by setting* $a_1 = 1$ *and*

$$a_{n+1} = 3 + \sqrt{a_n} \quad \text{for each } n \geq 1.$$

Show that $1 \leq a_n \leq a_{n+1} \leq 9$ *for each* $n \geq 1$*, use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 1$ and $a_2 = 3 + 1 = 4$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 9$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 1 \leq \sqrt{a_n} \leq \sqrt{a_{n+1}} \leq 3 \quad &\Longrightarrow \quad 4 \leq a_{n+1} \leq a_{n+2} \leq 6 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 9. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = 3 + \sqrt{a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = 3 + \lim_{n\to\infty} \sqrt{a_n} \quad \Longrightarrow \quad L = 3 + \sqrt{L}.$$

To solve this equation, we now set $x = \sqrt{L}$ to get

$$x^2 = 3 + x \quad \Longrightarrow \quad x^2 - x - 3 = 0 \quad \Longrightarrow \quad x = \frac{1 \pm \sqrt{13}}{2}.$$

Since $x = \sqrt{L}$ must be non-negative, only one of the roots is acceptable, and so

$$L = x^2 = \left(\frac{1+\sqrt{13}}{2}\right)^2 = \frac{1+13+2\sqrt{13}}{4} = \frac{7+\sqrt{13}}{2}.$$