MA121, Homework #4
Solutions

1. *Compute each of the following limits:*

$$L_1 = \lim_{x\to\infty} \frac{\log x}{x}, \qquad L_2 = \lim_{x\to 1} \frac{x^3 + x^2 - 5x + 3}{x^3 - x^2 - x + 1}.$$

Since the first limit gives ∞/∞, we may apply L'Hôpital's rule to get

$$L_1 = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Since the second limit gives $0/0$, L'Hôpital's rule is still applicable and we find

$$L_2 = \lim_{x\to 1} \frac{x^3 + x^2 - 5x + 3}{x^3 - x^2 - x + 1} = \lim_{x\to 1} \frac{3x^2 + 2x - 5}{3x^2 - 2x - 1}.$$

The last limit still gives $0/0$, so we may apply L'Hôpital's rule once again to get

$$L_2 = \lim_{x\to 1} \frac{3x^2 + 2x - 5}{3x^2 - 2x - 1} = \lim_{x\to 1} \frac{6x + 2}{6x - 2} = \frac{6+2}{6-2} = 2.$$

2. *Compute each of the following limits:*

$$M_1 = \lim_{x\to 0} \frac{3^x - 1}{x}, \qquad M_2 = \lim_{x\to 0} (e^x + x)^{1/x}.$$

Setting $x = 0$ in the first limit gives $0/0$, so we can apply L'Hôpital's rule to get

$$M_1 = \lim_{x\to 0} \frac{3^x - 1}{x} = \lim_{x\to 0} \frac{(3^x)'}{1} = \lim_{x\to 0} (3^x)'.$$

To compute the derivative of 3^x, we eliminate the exponent using logarithms, namely

$$\begin{aligned} f(x) = 3^x \quad &\Longrightarrow \quad \log f(x) = x \log 3 \quad \Longrightarrow \quad \frac{1}{f(x)} \cdot f'(x) = \log 3 \\ &\Longrightarrow \quad f'(x) = f(x) \cdot \log 3 = 3^x \log 3. \end{aligned}$$

Returning to our computation above, we may now conclude that

$$M_1 = \lim_{x\to 0} (3^x)' = \lim_{x\to 0} 3^x \log 3 = 3^0 \log 3 = \log 3.$$

To compute the second limit M_2, we eliminate the exponent using logarithms, namely

$$M_2 = \lim_{x\to 0} (e^x + x)^{1/x} \quad \Longrightarrow \quad \log M_2 = \lim_{x\to 0} \log(e^x + x)^{1/x} = \lim_{x\to 0} \frac{\log(e^x + x)}{x}.$$

Since the last limit gives $0/0$, we may then apply L'Hôpital's rule to get

$$\log M_2 = \lim_{x\to 0} \frac{(e^x + x)^{-1} \cdot (e^x + 1)}{1} = e^0 + 1 = 2$$

using simple substitution. This also implies that $M_2 = e^{\log M_2} = e^2$.

3. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{array} \right\}.$$

Show that f is integrable on $[0,1]$.

Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[0,1]$, one clearly has

$$S^-(f,P) = \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 0$$

because all the summands are zero. This also implies that $\sup S^-(f,P) = 0$ as well. To compute the infimum of the upper Darboux sums, we use a similar computation to get

$$S^+(f,P) = \sum_{k=0}^{n-1} \sup_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 1 \cdot (x_1 - x_0) = x_1.$$

This gives $\inf S^+(f,P) = \inf x_1 = 0 = \sup S^-(f,P)$, so f is integrable on $[0,1]$, indeed.

4. *Suppose f, g are both integrable on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$. Show that*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Starting with the inequality

$$f(x) \leq g(x) \quad \text{for all } \ x \in [x_k, x_{k+1}],$$

we may take the infimum of both sides to get

$$\inf_{[x_k,x_{k+1}]} f(x) \leq \inf_{[x_k,x_{k+1}]} g(x).$$

Multiplying by the positive quantity $x_{k+1} - x_k$ and then adding, we conclude that

$$\sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} g(x) \cdot (x_{k+1} - x_k).$$

Since the last inequality holds for all partitions P by above, we must actually have

$$S^-(f,P) \leq S^-(g,P)$$

for all partitions P. Taking the supremum of both sides, we finally deduce that

$$\int_a^b f(x)\,dx = \sup_P \{S^-(f,P)\} \ \leq \ \sup_P \{S^-(g,P)\} = \int_a^b g(x)\,dx.$$