MA121, Homework #3
Solutions

1. *Find the maximum value of* $f(x) = 3x^4 - 16x^3 + 18x^2$ *over the closed interval* $[-1, 2]$.

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-1)(x-3)$$

and the only points at which the maximum value may occur are

$$x = -1, \qquad x = 2, \qquad x = 0, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as this fails to lie in $[-1, 2]$, and we now compute

$$f(-1) = 37, \qquad f(2) = -8, \qquad f(0) = 0, \qquad f(1) = 5.$$

Based on these facts, we may finally conclude that the maximum value is $f(-1) = 37$.

2. *Show that the polynomial* $f(x) = x^3 - 3x + 1$ *has three roots in the interval* $(-2, 2)$. *As a hint, you might wish to compute the values of* f *at the points* ± 2, ± 1 *and* 0.

Being a polynomial, f is continuous on the closed interval $[-2, -1]$ and we also have

$$f(-2) = -1 < 0, \qquad f(-1) = 3 > 0.$$

Thus, f must have a root in $(-2, -1)$ by Bolzano's theorem. Using the facts that

$$f(0) = 1 > 0, \qquad f(1) = -1 < 0, \qquad f(2) = 3 > 0,$$

we similarly find that another root exists in $(0, 1)$ and that a third root exists in $(1, 2)$. In particular, f has three roots in $(-2, 2)$, as needed.

3. *Show that the polynomial* $f(x) = x^3 - 4x^2 - 3x + 1$ *has exactly one root in* $[0, 2]$.

Being a polynomial, f is continuous on the closed interval $[0, 2]$ and we also have

$$f(0) = 1 > 0, \qquad f(2) = -13 < 0.$$

Thus, f has a root in $(0, 2)$ by Bolzano's theorem and this root certainly lies in $[0, 2]$ as well. Suppose now that f has two roots in $[0, 2]$. By Rolle's theorem, f' must then have a root in $[0, 2]$ as well. On the other hand, the roots of $f'(x) = 3x^2 - 8x - 3$ are

$$x = \frac{8 \pm \sqrt{64 + 4 \cdot 3 \cdot 3}}{2 \cdot 3} = \frac{8 \pm 10}{6} \quad \Longrightarrow \quad x = 3, \quad x = -1/3.$$

Since neither of those lies in $[0, 2]$, we conclude that f cannot have two roots in $[0, 2]$.

4. *Apply the mean value theorem on the closed interval* $[1, 2]$ *to show that*

$$\frac{1}{2} < \log 2 < 1.$$

Since $f(x) = \log x$ is differentiable on $[1, 2]$, the mean value theorem gives

$$\frac{f(2) - f(1)}{2 - 1} = f'(c) \quad\Longrightarrow\quad \frac{\log 2 - \log 1}{2 - 1} = \frac{1}{c} \quad\Longrightarrow\quad \log 2 = \frac{1}{c}$$

for some $1 < c < 2$. Using the fact that $1 < c < 2$, one now finds that

$$1 > \frac{1}{c} > \frac{1}{2} \quad\Longrightarrow\quad 1 > \log 2 > \frac{1}{2}\,.$$

5. *Letting* $f(x) = 3x^4 - 16x^3 + 18x^2$ *for all* $x \in \mathbb{R}$, *compute each of the following:*

$$\max_{-1 \le x < 2} f(x), \qquad \max_{0 < x < 2} f(x), \qquad \min_{0 < x < 4} f(x), \qquad \min f(x).$$

• To find the minimum and maximum values of f over an arbitrary interval, one needs to first determine the sign of f' throughout the interval. In our case,

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x - 1)(x - 3)$$

and the sign of f' can be determined using the table below.

x		0		1		3	
$12x$	$-$		$+$		$+$		$+$
$x - 1$	$-$		$-$		$+$		$+$
$x - 3$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

• To find the maximum value of $f(x)$ when $-1 \le x < 2$, we need to compare

$$f(-1) = 37, \qquad f(1) = 5.$$

Since the former is bigger and also attained, this gives $\max_{-1 \le x < 2} f(x) = f(-1) = 37$.

• By our table above, the maximum value of $f(x)$ when $0 < x < 2$ is clearly $f(1) = 5$.

• To find the minimum value of $f(x)$ when $0 < x < 4$, we need to compare

$$f(0) = 0, \qquad f(3) = -27.$$

Since the latter is smaller and also attained, this gives $\min_{0 < x < 4} f(x) = f(3) = -27$.

• To find the minimum value of $f(x)$ over the whole real line, we need to compare

$$f(0) = 0, \qquad f(3) = -27.$$

Since the latter is smaller and also attained, this gives $\min f(x) = f(3) = -27$.