MA121, Homework #2
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \left\{n \in \mathbb{N} : \frac{n}{n+1} > \frac{3}{4}\right\}$

(b) $B = \{x \in \mathbb{R} : x > 1 \text{ and } 2x \leq 5\}$

(c) $C = \{x \in \mathbb{Z} : x > 1 \text{ and } 2x \leq 5\}$

(d) $D = \{x \in \mathbb{R} : x < y \text{ for all } y > 0\}$

(e) $E = \{x \in \mathbb{R} : 1 \leq |x - 2| < 3\}$

• A complete list of answers is provided by the following table.

	min	inf	max	sup
A	4	4	DNE	DNE
B	DNE	1	5/2	5/2
C	2	2	2	2
D	DNE	DNE	0	0
E	DNE	-1	DNE	5

• The set A contains all $n \in \mathbb{N}$ with $4n > 3n + 3$; this means that $A = \{4, 5, 6, \ldots\}$.

• The set B contains all $x \in \mathbb{R}$ with $1 < x \leq 5/2$; this means that $B = (1, 5/2]$.

• The set C contains all integers x with $1 < x \leq 5/2$; this means that $C = \{2\}$.

• The set D contains the real numbers x which are smaller than all positive reals; this means that $D = (-\infty, 0]$.

• The set E contains the real numbers x whose distance from 2 is at least 1 but strictly less than 3; a quick sketch should convince you that $E = (-1, 1] \cup [3, 5)$.

2. *Let $x \in \mathbb{R}$ be such that $x > -1$. Show that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.*

• We use induction to prove the given inequality for all $n \in \mathbb{N}$.

• When $n = 1$, the given inequality holds because $(1 + x)^1 = 1 + x = 1 + nx$.

• Suppose that the inequality holds for some n, in which case

$$(1 + x)^n \geq 1 + nx.$$

Since $1 + x > 0$ by assumption, we may then multiply this inequality by $1 + x$ to get

$$(1 + x)^{n+1} \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$$

because $nx^2 \geq 0$. This actually proves the given inequality for $n + 1$, as needed.

3. *Let A, B be nonempty subsets of $\mathbb{R}$ such that $\inf A < \inf B$. Show that there exists an element $a \in A$ which is a lower bound of B.*

Since $\inf B$ is bigger than the greatest lower bound of A, we see that $\inf B$ cannot be a lower bound of A. This means that some element $a \in A$ is such that $a < \inf B$. Using the fact that $\inf B$ is a lower bound of B, we now find that $a < \inf B \leq b$ for all $b \in B$. This means that a itself is a lower bound of B.

4. *Evaluate the limit*

$$L = \lim_{x \to 1} \frac{6x^3 - 13x^2 + 4x + 3}{x - 1}.$$

Using division of polynomials, one easily finds that

$$L = \lim_{x \to 1} \frac{6x^3 - 13x^2 + 4x + 3}{x - 1} = \lim_{x \to 1} (6x^2 - 7x - 3) = 6 - 7 - 3 = -4$$

since $x \neq 1$ and since limits of polynomials can be computed by simple substitution.

5. *Let f be a function such that $|f(x) - 3| \leq 5|x|$ for all $x \in \mathbb{R}$. Show that $\lim_{x \to 0} f(x) = 3$.*

Let $\varepsilon > 0$ be given and set $\delta = \varepsilon/5$. Then $\delta > 0$ and we easily find that

$$0 \neq |x - 0| < \delta \quad \Longrightarrow \quad |f(x) - 3| \leq 5|x| < 5\delta = \varepsilon.$$

6. *Show that the function f defined by*

$$f(x) = \left\{ \begin{array}{ll} 3x - 2 & \text{if } x \leq 2 \\ 4x - 4 & \text{if } x > 2 \end{array} \right\}$$

is continuous at $y = 2$.

To prove that f is continuous at $y = 2$, let us first note that

$$|f(x) - f(2)| = |f(x) - 4| = \left\{ \begin{array}{ll} 3|x - 2| & \text{if } x \leq 2 \\ 4|x - 2| & \text{if } x > 2 \end{array} \right\}.$$

Now, let $\varepsilon > 0$ be given and set $\delta = \varepsilon/4$. Then $\delta > 0$ and we easily find that

$$|x - 2| < \delta \quad \Longrightarrow \quad |f(x) - f(2)| \leq 4|x - 2| < 4\delta = \varepsilon.$$