MA121, Homework #1
Solutions

1. *Show that the product of two negative real numbers is positive.*

Suppose that $x < 0$ and $y < 0$. Since multiplication by a negative number reverses the order of inequalities, we must then have $-x > 0$ and $-y > 0$. This implies that

$$(-x)(-y) > 0$$

because the product of two positive real numbers is positive by one of our axioms. Thus,

$$(-x)(-y) > 0 \quad \Longrightarrow \quad xy > 0$$

because the two negative signs cancel by one of our lemmas.

2. *Show that $x + \frac{1}{x} \geq 2$ for every real number $x > 0$.*

Since $x > 0$ by assumption, the given inequality is equivalent to

$$\begin{aligned} x + \frac{1}{x} \geq 2 \quad &\Longleftrightarrow \quad x^2 + 1 \geq 2x \quad \Longleftrightarrow \quad x^2 - 2x + 1 \geq 0 \\ &\Longleftrightarrow \quad (x-1)^2 \geq 0. \end{aligned}$$

Note that the last inequality is certainly true because squares are always non-negative. Thus, the first inequality must also be true, as needed.

3. *Show that the set $A = \{x \in \mathbb{R} : |2x - 3| < 1\}$ has no maximum.*

First of all, note that the given inequality is equivalent to

$$\begin{aligned} |2x - 3| < 1 \quad &\Longleftrightarrow \quad -1 < 2x - 3 < 1 \quad \Longleftrightarrow \quad 2 < 2x < 4 \\ &\Longleftrightarrow \quad 1 < x < 2. \end{aligned}$$

Suppose now that $x \in A$. Then $1 < x < 2$ and so the average $z = \frac{x+2}{2}$ is such that

$$1 < x < z < 2.$$

This shows that z is an element of A which is bigger than x, namely that x is not the maximum element of A. In particular, A does not have a maximum at all.

4. *Show that the set $B = \{-2x^2 + 3x : x \in \mathbb{R}\}$ has a maximum and find it explicitly.*

To determine the maximum of the given set, let us first note that

$$-2x^2 + 3x = -2\left(x^2 - \frac{3x}{2} + \frac{9}{16} - \frac{9}{16}\right) = -2(x - 3/4)^2 + 9/8 \leq 9/8$$

and that equality holds in the last inequality when $x = 3/4$. This makes $9/8$ an element of B which is at least as large as any other element of B, hence $\max B = 9/8$.

5. *Show that the set* $C = \{\frac{n+1}{n} : n \in \mathbb{N}\}$ *is such that* $\inf C = 1$.

To see that 1 is a lower bound of the given set, we note that

$$n \in \mathbb{N} \quad \Longrightarrow \quad n + 1 > n \quad \Longrightarrow \quad \frac{n+1}{n} > 1.$$

To see that 1 is the greatest lower bound, suppose that $x > 1$ and note that

$$\frac{n+1}{n} < x \quad \Longleftrightarrow \quad n + 1 < nx \quad \Longleftrightarrow \quad 1 < n(x-1) \quad \Longleftrightarrow \quad \frac{1}{x-1} < n.$$

According to one of our theorems, we can always find an integer $n \in \mathbb{N}$ such that $n > \frac{1}{x-1}$. Then our computation above shows that $\frac{n+1}{n} < x$. In particular, x is strictly larger than an element of C, so x cannot possibly be a lower bound of C.

6. *Show that the set* $D = \{x \in \mathbb{R} : x^2 \geq x\}$ *has no supremum.*

It is easy to see that the set D contains all the natural numbers $n \in \mathbb{N}$, namely

$$n \in \mathbb{N} \quad \Longrightarrow \quad n \geq 1 \quad \Longrightarrow \quad n^2 \geq n \quad \Longrightarrow \quad n \in D.$$

Suppose now that $\sup D$ actually exists. Then it must be the case that

$$x \leq \sup D \quad \text{for all } x \in D.$$

Since every natural number is an element of D by above, however, this also implies

$$n \leq \sup D \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\sup D$ is an upper bound of $\mathbb{N}$, violating the fact that $\mathbb{N}$ has no upper bound.