MA121, Homework #8
due Monday, Apr. 28 in class

1. Letting $f(x, y) = \log(x^2 + y^2)$, find the rate at which f is changing at the point $(1, 2)$ in the direction of the vector $\mathbf{v} = \langle 6, 8 \rangle$.

2. Assuming that $z = f(x^2 - y^2, y^2 - x^2)$ for some differentiable function f, show that

$$yz_x + xz_y = 0.$$

3. Find the maximum value of $f(x, y) = x^2 + xy + 3x + 2y$ over the region

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 \le y \le 4\}.$$

4. Classify the critical points of $f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2$.

5. Classify the critical points of $f(x, y) = 3x^2 - 3y^2 + 8xy + 10x - 20y + 30$.

- You are going to work on these problems during your Friday tutorials.

- When writing up solutions, write legibly and coherently. Use words, not just symbols.

- Write both your name and your tutor's name on the first page of your homework.

- Your tutor's name is Derek, if you are a TP student; otherwise, it is Pete.

- Your solutions may use any of the axioms/results stated in class (but nothing else).

- NO LATE HOMEWORK WILL BE ACCEPTED.

Hints and comments

1. To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, we need to divide $\mathbf{v}$ by its length, namely

$$||\mathbf{v}|| = \sqrt{6^2 + 8^2} = 10 \quad \Longrightarrow \quad \mathbf{u} = \frac{1}{10}\mathbf{v} = \langle 6/10, 8/10 \rangle .$$

   The desired rate of change is given by the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Since

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle \quad \Longrightarrow \quad \nabla f(1, 2) = \langle 2/5, 4/5 \rangle ,$$

   we may thus conclude that the desired rate of change is $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 22/25$.

2. In this case, $z = f(u, v)$ with $u = x^2 - y^2$ and $v = y^2 - x^2$. You need to compute

$$z_x = f_x = f_u u_x + f_v v_x, \qquad z_y = f_y = f_u u_y + f_v v_y.$$

3. Draw a sketch to convince yourselves that R is both closed and bounded. There is only one critical point, but it fails to lie in the region so you need to ignore it. The boundary consists of two parts to be treated separately. Find the maximum value of

$$g(x) = f(x, x^2) = x^3 + 3x^2 + 3x$$

   over the closed interval $[-2, 2]$ and do the same thing for $h(x) = f(x, 4) = x^2 + 7x + 8$. Do this carefully because there is a critical point that fails to lie in $[-2, 2]$.

4. To find the critical points, you need to solve the equations

$$0 = f_x(x, y) = 4x(x^2 + 2y^2 - 1), \qquad 0 = f_y(x, y) = 4y(2x^2 + 1).$$

   Note that the second equation gives $y = 0$ and then simplify. The critical points are

$$(0, 0), \qquad (1, 0), \qquad (-1, 0).$$

   As it turns out, the last two are local minima, while the origin is a saddle point.

5. To find the critical points, you need to solve the equations

$$0 = f_x(x, y) = 6x + 8y + 10, \qquad 0 = f_y(x, y) = -6y + 8x - 20.$$

   As it turns out, the only critical point is $(1, -2)$ and this is a saddle point.