MA121, Homework #7

due Monday, Apr. 14 in class

1. Compute each of the following sums in terms of known functions:

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!}\,, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, 4^n x^{2n+1}}{(2n+1)!}\,, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{9^n \cdot (2n)!}\,.$$

2. Show that the equation $r = 3\sin\theta$ describes a circle in polar coordinates. As a hint, you may wish to multiply this equation by r and then express it in terms of x and y.

3. Compute the area of a right triangle whose sides have length a, b and $\sqrt{a^2+b^2}$.

4. Let R be the region between the graph of $f(x) = e^x - 1$ and the x-axis on $[0, 1]$. Find the volume of the solid obtained upon rotation of R around the x-axis.

5. Compute each of the following limits:

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2}\,, \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}\,, \qquad \lim_{(x,y)\to(1,1)} \frac{3x^2 - xy - 2y^2}{x-y}\,.$$

- You are going to work on these problems during your Friday tutorials.
- When writing up solutions, write legibly and coherently. Use words, not just symbols.
- Write both your name and your tutor's name on the first page of your homework.
- Your tutor's name is Derek, if you are a TP student; otherwise, it is Pete.
- Your solutions may use any of the axioms/results stated in class (but nothing else).
- NO LATE HOMEWORK WILL BE ACCEPTED.

Hints and comments

1. In each case, you are supposed to manipulate some of the known Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}.$$

When it comes to the first sum, for instance, one can easily see that

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!} = x \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = xe^{-x^2}.$$

2. Following the hint, multiply by r and switch to rectangular coordinates. This gives

$$r = 3\sin\theta \quad\Longrightarrow\quad r^2 = 3r\sin\theta \quad\Longrightarrow\quad x^2 + y^2 = 3y.$$

Move all terms on one side and then complete the square: $x^2 + (y - 3/2)^2 = (3/2)^2$.

3. Let $f(x) = bx/a$ for all $x \in [0, a]$. Then the desired triangle is merely the region that lies between the graph of f and the x-axis, so its area is given by

$$\text{Area} = \int_0^a f(x)\, dx = \int_0^a \frac{bx}{a}\, dx.$$

4. According to our formula, the desired volume is given by

$$\text{Volume} = \int_0^1 \pi f(x)^2\, dx = \pi \int_0^1 (e^x - 1)^2\, dx = \pi \int_0^1 (e^{2x} - 2e^x + 1)\, dx.$$

5a. Since rational functions are continuous, you may substitute $x = 1$ and $y = 2$.

5b. Switch to polar coordinates and note that

$$f(x, y) = \frac{x^2 y}{x^2 + y^2} = r\cos^2\theta \sin\theta.$$

Since (x, y) approaches the origin, we have $r = \sqrt{x^2 + y^2} \to 0$ and so the given function must approach zero as well. Work this out carefully using the Squeeze Law:

$$0 \le |f(x, y)| = |r\cos^2\theta \sin\theta| \le r.$$

5c. Use division of polynomials to simplify the given fraction. In case it helps, try to treat y as a constant and thus view the numerator/denominator as polynomials in x.