MA121, Homework #6

due Monday, Feb. 25 in class

1. Test each of the following series for convergence:

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}\,, \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n}\,, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, e^{1/n}}{n}\,.$$

2. Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}\,, \qquad \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}\,.$$

3. Differentiate the formula for a geometric series to show that

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \qquad \text{whenever } |x| < 1.$$

4. Use the nth term test to show that each of the following series diverges:

$$\sum_{n=1}^{\infty} n^{1/n}, \qquad \sum_{n=1}^{\infty} n\sin(1/n).$$

- You are going to work on these problems during your Friday tutorials.
- When writing up solutions, write legibly and coherently. Use words, not just symbols.
- Write both your name and your tutor's name on the first page of your homework.
- Your tutor's name is Derek, if you are a TP student; otherwise, it is Pete.
- Your solutions may use any of the axioms/results stated in class (but nothing else).
- NO LATE HOMEWORK WILL BE ACCEPTED.

Hints and comments

1a. Since $1/n \leq 1$, we have $e^{1/n} \leq e$. Use the comparison test to compare with $b_n = e/n^2$.

1b. Use the limit comparison test to compare with $b_n = 1/n$. Here, the point is that

$$\lim_{n\to\infty} e^{1/n} = e^0 = 1.$$

1c. Use the alternating series test. In this case, you have to check that

$$a_n = \frac{e^{1/n}}{n}$$

is non-negative, decreasing and approaching zero as $n \to \infty$. The hardest part is to check that a_n is decreasing; use derivatives in order to check this.

2a. One always determines the radius of convergence using the ratio test. In this case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \frac{|x|}{3}$$

so the series converges when $|x|/3 < 1$ and diverges when $|x|/3 > 1$. In other words, the series converges when $|x| < 3$ and diverges when $|x| > 3$. This also means that $R = 3$.

2b. Following the argument of part (a), you should find that $R = 1$.

3. Since $|x| < 1$ by assumption, the formula for a geometric series is applicable and it gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}.$$

Differentiate both sides of this equation and then multiply by x.

4a. You have to show that $n^{1/n}$ fails to approach zero as $n \to \infty$. Thus, you need to compute

$$L = \lim_{n\to\infty} n^{1/n} \quad \Longrightarrow \quad \log L = \lim_{n\to\infty} \frac{\log n}{n}.$$

The rightmost limit is an ∞/∞ limit, so you can use L'Hôpital's rule.

4b. Once again, you have to compute a certain limit, namely

$$L = \lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}.$$

The limit on the left is an $\infty \cdot 0$ limit, whereas the limit on the right is a $0/0$ one. This means that you can use L'Hôpital's rule to compute the latter.