MA121, Homework #5

due Monday, Feb. 11 in class

1. Compute each of the following integrals:

$$\int 2x^3 e^{x^2}\,dx, \qquad \int \frac{x^2+3}{x(x+1)^2}\,dx.$$

2. Compute each of the following integrals:

$$\int (\log x)^2\,dx, \qquad \int \frac{x^2+x-3}{x^2+3x+2}\,dx.$$

3. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and

$$a_{n+1} = 3 + \sqrt{a_n} \quad \text{for each } n \geq 1.$$

Show that $1 \leq a_n \leq a_{n+1} \leq 9$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

- You are going to work on these problems during your Friday tutorials.
- When writing up solutions, write legibly and coherently. Use words, not just symbols.
- Write both your name and your tutor's name on the first page of your homework.
- Your tutor's name is Derek, if you are a TP student; otherwise, it is Pete.
- Your solutions may use any of the axioms/results stated in class (but nothing else).
- NO LATE HOMEWORK WILL BE ACCEPTED.

Hints and comments

1a. Use the substitution $u = x^2$. This gives $du = 2x\,dx$, hence also

$$\int 2x^3 e^{x^2}\,dx = \int 2x \cdot x^2 e^{x^2}\,dx = \int u e^u\,du.$$

To compute the last integral, use either tabular integration or integration by parts.

1b. Use partial fractions to write

$$\frac{x^2+3}{x(x+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x+1)^2}$$

for some constants A, B, C. You will eventually need to use the substitution $u = x+1$.

2a. Integrate by parts to find that

$$\int (\log x)^2\,dx = \int x'(\log x)^2\,dx = x(\log x)^2 - 2\int \log x\,dx.$$

To compute the last integral, integrate by parts once again.

2b. Using division of polynomials and then partial fractions, one can write

$$\frac{x^2+x-3}{x^2+3x+2} = 1 - \frac{2x+5}{x^2+3x+2} = 1 + \frac{A}{x+1} + \frac{B}{x+2}.$$

Once the constants A, B have been determined, you can easily integrate term by term.

3. Use induction to show that $1 \leq a_n \leq a_{n+1} \leq 9$ for each $n \geq 1$. This makes the sequence monotonic and bounded, so it must be convergent. Denote its limit by L and argue that

$$a_{n+1} = 3 + \sqrt{a_n} \quad\Longrightarrow\quad L = 3 + \sqrt{L}.$$

To solve this equation, you need to note that it has the form $x^2 = 3 + x$ for some x.