MA121, Homework #3

due Monday, Dec. 3 in class

1. Find the maximum value of $f(x) = 3x^4 - 16x^3 + 18x^2$ over the closed interval $[-1, 2]$.

2. Show that the polynomial $f(x) = x^3 - 3x + 1$ has three roots in the interval $(-2, 2)$. As a hint, you might wish to compute the values of f at the points ± 2, ± 1 and 0.

3. Show that the polynomial $f(x) = x^3 - 4x^2 - 3x + 1$ has exactly one root in $[0, 2]$.

4. Apply the mean value theorem on the closed interval $[1, 2]$ to show that

$$\frac{1}{2} < \log 2 < 1.$$

5. Letting $f(x) = 3x^4 - 16x^3 + 18x^2$ for all $x \in \mathbb{R}$, compute each of the following:

$$\max_{-1 \leq x < 2} f(x), \qquad \max_{0 < x < 2} f(x), \qquad \min_{0 < x < 4} f(x), \qquad \min f(x).$$

- You are going to work on these problems during your Friday tutorials.
- When writing up solutions, write legibly and coherently. Use words, not just symbols.
- Write both your name and your tutor's name on the first page of your homework.
- Your tutor's name is Derek, if you are a TP student; otherwise, it is Pete.
- Your solutions may use any of the axioms/results stated in class (but nothing else).

Some hints

1. Since you are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In this case,

$$f'(x) = 12x(x-1)(x-3).$$

Make sure that you only consider points which lie on the given closed interval.

2. Use Bolzano's theorem to find a root in $(-2,-1)$, a root in $(0,1)$ and one in $(1,2)$.

3. The fact that there exists a root in $[0,2]$ follows by Bolzano's theorem. Suppose now that there exist two roots in $[0,2]$. Then $f'(x)$ must have a root in $[0,2]$ by Rolle's theorem. Find the roots of $f'(x)$ to see that this is not the case.

4. Since $f(x) = \log x$ is differentiable on $[1,2]$, the mean value theorem gives

$$\frac{f(2)-f(1)}{2-1} = f'(c) \quad\Longrightarrow\quad \frac{\log 2 - \log 1}{2-1} = \frac{1}{c} \quad\Longrightarrow\quad \log 2 = \frac{1}{c}$$

for some $1 < c < 2$. Use the fact that $1 < c < 2$ to now estimate $\log 2$.

5. You will need to determine the sign of f'. In this case,

$$f'(x) = 12x(x-1)(x-3)$$

and the sign of f' can be determined using the table below.

x		0	1	3
$12x$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

To find the maximum value when $-1 \leq x < 2$, for instance, one needs to compare

$$f(-1) = 37, \qquad f(1) = 5.$$

Since the former is bigger and also attained, this gives $\max_{-1\leq x<2} f(x) = f(-1) = 37$.