Chapter 5

Sequences and series

5.1 Sequences

Definition 5.1 (Sequence). A sequence is a function which is defined on the set $\mathbb{N}$ of natural numbers. Since such a function is uniquely determined by its values $f(1)$, $f(2)$ and so on, it is usually denoted by writing $a_n = f(n)$ for each $n \in \mathbb{N}$.

Definition 5.2 (Convergence). A sequence $\{a_n\}$ is called convergent, if $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 5.3 (A useful limit). Given any real number x, one has

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Notation (Factorial). We denote by $n! = 1 \cdot 2 \cdot \ldots \cdot n$ the product of the first n positive integers, and we also use the convention that $0! = 1$.

Definition 5.4 (Monotonic). A sequence $\{a_n\}$ is called monotonic, if it is either increasing, in which case $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, or decreasing, in which case $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Example 5.5. To check that the sequence $a_n = \frac{n}{n+1}$ is increasing, we define the function

$$f(x) = \frac{x}{x+1}\,, \qquad x \geq 1$$

and we check that f is increasing, instead. Using the quotient rule, we easily find that

$$f'(x) = \frac{1 \cdot (x+1) - 1 \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0.$$

Thus, $f(x)$ is increasing for all $x \geq 1$, and this forces a_n to be increasing for all $n \geq 1$.

Example 5.6. To check that the sequence $a_n = \frac{2^n}{n!}$ is decreasing, we cannot really follow our previous approach since $n!$ is only defined when n is a non-negative integer. In this case, it is better to look at the ratio of two consecutive terms, namely at the ratio

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \frac{2}{n+1}\,.$$

If this ratio is less than 1, then $a_{n+1} \leq a_n$ and so the sequence is decreasing. In our case,

$$\frac{a_{n+1}}{a_n} \leq 1 \quad\iff\quad \frac{2}{n+1} \leq 1 \quad\iff\quad 2 \leq n+1$$

and so the sequence is decreasing for each $n \geq 1$.

Theorem 5.7 (Convergent implies bounded). Every convergent sequence is bounded.

Theorem 5.8 (Monotonic and bounded). If a sequence is both monotonic and bounded, then it must necessarily converge.

Example 5.9. Consider the sequence $\{a_n\}$ which is defined by setting $a_1 = 1$ and

$$a_{n+1} = \sqrt{2a_n} \qquad \text{for each } n \geq 1.$$

To show that this sequence converges, we shall first show that

$$1 \leq a_n \leq a_{n+1} \leq 2 \qquad \text{for each } n \geq 1. \tag{$*$}$$

When $n = 1$, this statement asserts that $1 \leq 1 \leq \sqrt{2} \leq 2$, so it is clearly true. Suppose the statement holds for some n. Multiplying by 2 and taking square roots, we then find that

$$\begin{aligned} \sqrt{2} \leq \sqrt{2a_n} \leq \sqrt{2a_{n+1}} \leq 2 \quad &\implies \quad \sqrt{2} \leq a_{n+1} \leq a_{n+2} \leq 2 \\ &\implies \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 2. \end{aligned}$$

In particular, our statement $(*)$ holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{2a_n} \quad\implies\quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} \quad\implies\quad L = \sqrt{2L}.$$

This gives $L^2 = 2L$, so either $L = 0$ or else $L = 2$. On the other hand, we must also have

$$1 \leq a_n \leq 2 \quad\implies\quad 1 \leq \lim_{n\to\infty} a_n \leq 2 \quad\implies\quad 1 \leq L \leq 2$$

because of the statement $(*)$ we just proved. Thus, the limit of the sequence is $L = 2$.

Lemma 5.10. Given any real number x with $|x| < 1$, one has $\lim_{n\to\infty} x^n = 0$.

Lemma 5.11. Every sequence has a monotonic subsequence.

Theorem 5.12 (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

5.2 Infinite series

Definition 5.13 (Partial sums). Given a sequence $\{a_n\}$, we define the sequence $\{s_n\}$ of its partial sums by setting $s_n = a_1 + a_2 + \ldots + a_n$ for each $n \in \mathbb{N}$.

Definition 5.14 (Infinite series). Let $\{a_n\}$ be a given sequence and $\{s_n\}$ the sequence of its partial sums. In the case that $\{s_n\}$ happens to converge, we introduce the series

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$

and we say that this series converges. Otherwise, we say that the series diverges.

Proposition 5.15 (Properties of series). Let $\{a_n\}$, $\{b_n\}$ be sequences and let $c \in \mathbb{R}$.

(a) One has $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ as long as the two series on the right converge.

(b) One has $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$ as long as the series on the right converges.

Theorem 5.16 (nth term test). If the series $\sum_{n=1}^{\infty} a_n$ converges, then it must be the case that

$$\lim_{n\to\infty} a_n = 0.$$

In other words, the series diverges whenever its nth term fails to approach zero as $n \to \infty$.

Example 5.17. As one can easily see, each of the following series is divergent:

$$\sum_{n=1}^{\infty} (-1)^n, \qquad \sum_{n=1}^{\infty} n, \qquad \sum_{n=1}^{\infty} \frac{n+1}{n}, \qquad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 5.18 (Geometric series). Let $x \in \mathbb{R}$ be fixed. Then the geometric series $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$, in which case we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Example 5.19. We use the last formula to explicitly compute the sum

$$S = \sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n+1}}.$$

Let us first isolate the part of the exponents which does not depend on n; we get

$$S = \sum_{n=1}^{\infty} \frac{2^n \cdot 4}{3^{2n} \cdot 3} = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \frac{2^n}{9^n} = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n.$$

Since the rightmost sum is a geometric series without its first term, this actually gives

$$S = \frac{4}{3} \cdot \left[\sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n - 1\right] = \frac{4}{3} \cdot \left[\frac{1}{1-2/9} - 1\right] = \frac{4}{3} \cdot \left[\frac{9}{7} - 1\right] = \frac{8}{21}.$$

5.3 Tests for convergence

Lemma 5.20 (Non-negative series). Let $\{a_n\}$ be a sequence of non-negative terms. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is bounded.

Lemma 5.21 (Integral Test). Suppose f is non-negative and decreasing on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral

$$\int_1^n f(x)\, dx$$

is bounded for all $n \in \mathbb{N}$. We shall only use this test in order to prove the following theorem.

Theorem 5.22 (p-series). The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Theorem 5.23 (Comparison Test). Suppose that $\{a_n\}$ and $\{b_n\}$ are non-negative with

$$0 \leq a_n \leq b_n \qquad \text{for all } n \in \mathbb{N}.$$

If the series $\sum_{n=1}^{\infty} b_n$ happens to converge, then the series $\sum_{n=1}^{\infty} a_n$ must also converge. And if the series $\sum_{n=1}^{\infty} a_n$ happens to diverge, then the series $\sum_{n=1}^{\infty} b_n$ must also diverge. In short, smaller than convergent implies convergent and bigger than divergent implies divergent.

Example 5.24. We use the comparison test to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n}$$

is convergent. Since the denominator is at least as large as 2^n, it is clear that

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n.$$

Being smaller than a convergent geometric series, the given series must thus be convergent by the comparison test. Needless to say, one could also argue that

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n} \leq \sum_{n=1}^{\infty} \frac{1}{n},$$

however this inequality does not help because the rightmost series is a divergent p-series.

Theorem 5.25 (Limit Comparison Test). Suppose $\{a_n\}$ and $\{b_n\}$ are non-negative with

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges.

Remark. The limit comparison test is especially useful when a_n is a rational function. In that case, ignoring the lower-order terms in both the numerator and the denominator of a_n gives rise to a simple rational function b_n for which the limit comparison test applies.

Example 5.26. We use the limit comparison test to check the series

$$\sum_{n=1}^{\infty} \frac{2n^2+n+1}{n^3+2}$$

for convergence. Let a_n denote its nth term, which we are going to compare with

$$b_n = \frac{2n^2}{n^3} = \frac{2}{n}\,.$$

To see that a_n and b_n are roughly the same thing when n is large, we compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n+1}{n^3+2} \cdot \frac{n}{2} = \lim_{n\to\infty} \frac{2n^3+n^2+n}{2n^3+4} = 1.$$

By the limit comparison test, the series corresponding to a_n converges if and only if the one corresponding to b_n does. Since the latter is a divergent p-series, they must both diverge.

Theorem 5.27 (Ratio Test). Let $\sum_{n=1}^{\infty} a_n$ be a given series and consider the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, then the given series converges. And if $L > 1$, then the given series diverges.

Remark. The ratio test provides no conclusions for the remaining case $L = 1$. Should that case arise, one has to apply some other test for convergence. The ratio test is especially useful when a_n involves either exponents or factorials.

Example 5.28. We use the ratio test to check the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

for convergence. In this case, the ratio of two consecutive terms has limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}\,.$$

Since this is strictly smaller than 1, the given series converges by the ratio test.

Definition 5.29 (Absolute convergence). A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent, if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem 5.30 (Absolute convergence). Absolute convergence implies convergence: if the series with the absolute values converges, then the series without them converges as well.

Example 5.31. Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

If we replace the nth term by its absolute value, then we end up with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is known to converge. This shows that the given series converges absolutely. Using the last theorem, we conclude that the original series converges as well.

Theorem 5.32 (Limits and inequalities). Suppose $\{a_n\}$ and $\{b_n\}$ are convergent with

$$a_n \leq b_n \qquad \text{for all } n \geq 1.$$

Then one may take limits of both sides to conclude that

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Theorem 5.33 (Alternating Series Test). Suppose $\{a_n\}$ is non-negative, decreasing with

$$\lim_{n\to\infty} a_n = 0.$$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ must necessarily converge.

Example 5.34. To show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{n^2+1}$ converges, we need only check that the function

$$f(x) = \frac{x}{x^2+1}, \qquad x \geq 1$$

is decreasing to zero. Now, the fact that f is decreasing follows by the quotient rule since

$$f'(x) = \frac{x^2+1-2x\cdot x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$$

whenever $x \geq 1$. As for the fact that f is decreasing to zero, this follows by the computation

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1/x}{1+1/x^2} = \frac{0}{1+0} = 0.$$

5.4 Power series

Definition 5.35 (Power series). A power series is a sum of powers of x such as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

for some coefficients $a_n \in \mathbb{R}$. As a function of x, this may only be defined for the values of x for which the series converges; one always uses the ratio test to determine those values.

Definition 5.36 (Radius of convergence). If a power series converges when $|x| < R$ and diverges when $|x| > R$, then we call R the radius of convergence.

Example 5.37. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$. In this case, we have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n+1}{n+2} = |x| \cdot \lim_{n\to\infty} \frac{n+1}{n+2} = |x|.$$

In view of the ratio test then, the series converges when $|x| < 1$ and it diverges when $|x| > 1$. In particular, its radius of convergence is equal to $R = 1$.

Example 5.38. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case, we have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since this is smaller than 1, the series converges by the ratio test (for any x whatsoever).

Theorem 5.39 (Differentiation of power series). Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges when $|x| < R$. Then the power series

$$g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

converges when $|x| < R$ as well, and we also have $f'(x) = g(x)$ for all such x. In other words, one may differentiate a power series by differentiating it term by term.

Notation (Higher derivatives). The nth derivative of a given function $f(x)$ is usually denoted by $f^{(n)}(x)$. In particular, one has $f^{(1)}(x) = f'(x)$ and $f^{(0)}(x) = f(x)$ by convention.

Definition 5.40 (Taylor series and polynomials). Suppose f is differentiable an infinite number of times. Then its Taylor series (around the point $x = 0$) is defined to be the series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n.$$

The Nth Taylor polynomial of f is defined as a truncated version of this series, namely

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \cdot x^n.$$

Theorem 5.41 (Taylor's theorem, integral form). Suppose f is differentiable an infinite number of times. Then the difference between f and its Nth Taylor polynomial is given by

$$f(x) - T_N(x) = \int_0^x \frac{f^{(N+1)}(t)}{N!} \cdot (x-t)^N \, dt.$$

Theorem 5.42 (Taylor's theorem, differential form). Suppose f is differentiable an infinite number of times. Then there exists a number c between 0 and x such that

$$f(x) - T_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} \cdot x^{N+1}.$$

Remark. To show that a function $f(x)$ is equal to its Taylor series, one has to show that

$$0 = f(x) - T(x) = \lim_{N\to\infty} [f(x) - T_N(x)].$$

The last two theorems are useful because they allow us to simplify the right hand side.

Theorem 5.43 (sin and cos). There exists a unique pair of functions $\sin x, \cos x$ such that

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin 0 = 0, \qquad \cos 0 = 1.$$

Moreover, these functions are defined for all $x \in \mathbb{R}$ and they have the following properties:

(a) $\sin x$ is an odd function in the sense that $\sin(-x) = -\sin x$ for all $x \in \mathbb{R}$;

(b) $\cos x$ is an even function in the sense that $\cos(-x) = \cos x$ for all $x \in \mathbb{R}$;

(c) $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$, hence $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all $x \in \mathbb{R}$.

Theorem 5.44 (Known Taylor series). One has the formulas

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$, as well as the formula

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \text{whenever } |x| < 1.$$

Application 5.45 (Other Taylor series). Using the formulas above, one can also compute the Taylor series for related functions. For instance, the Taylor series of $f(x) = x^3 \sin(2x)$ is

$$f(x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+4}}{(2n+1)!}$$

and the Taylor series of $g(x) = e^{1-x}$ is

$$g(x) = ee^{-x} = e \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{e(-1)^n x^n}{n!}.$$

Remark (Shifting indices). A series can be written in many different ways using the sigma notation. For instance, it should be easy to see that the expressions

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{n=2}^{\infty} a_{n-1}, \qquad \sum_{n=4}^{\infty} a_{n-3}$$

are all equal. As a general rule for shifting the index of summation, one gets to increase the values of n in the index while decreasing the values of n in the summand (and vice versa).

Example 5.46. Using the formula for a geometric series, one easily finds that

$$\sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1} = x \sum_{n=0}^{\infty} x^n = \frac{x}{1-x} \qquad \text{whenever } |x| < 1.$$

Application 5.47 (Computing sums). Several infinite series can be computed explicitly by reducing them to the Taylor series of a known function. For instance, we can compute

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n!} = \sum_{n=1}^{\infty} \frac{(-4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} - 1 = e^{-4} - 1$$

using the Taylor series for the exponential function, and we can similarly compute

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^{n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n+2}}{(2n+1)!} = 2\left(\sin 2 - \frac{2^1}{1!}\right) = 2\sin 2 - 4.$$

Theorem 5.48 (Binomial series). Given any real number α, one has the formula

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$$

whenever $|x| < 1$. In the special case that $\alpha = n$ is a positive integer, this formula reads

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots + nx^{n-1} + x^n$$

and it actually holds for any $x \in \mathbb{R}$ whatsoever.

Application 5.49 (Approximations). The Taylor series around the point $x = 0$ expresses a given function as a sum of powers of x. When x is sufficiently small, however, the higher-order powers of x are negligible and can thus be ignored when seeking an approximation to a given function. For instance, the Taylor series for the sine function is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

and this leads to the approximation $\sin x \approx x - x^3/6$ for all small enough x.

Example 5.50. We use the 3rd Taylor polynomial for the sine function to show that $\sin 1$ is positive. By above, the 3rd Taylor polynomial of $f(x) = \sin x$ is

$$T_3(x) = x - \frac{x^3}{6} \quad \Longrightarrow \quad T_3(1) = 1 - \frac{1}{6} = \frac{5}{6}.$$

The value $T_3(1) = 5/6$ is not really the actual value of $f(1) = \sin 1$, however we do have

$$f(1) - T_3(1) = \frac{f^{(4)}(c)}{4!} = \frac{f^{(4)}(c)}{24}$$

for some $c \in (0, 1)$ by Taylor's theorem. Since $f^{(4)}(x) = \sin x$ for all x, this gives

$$-\frac{1}{24} \leq f(1) - T_3(1) \leq \frac{1}{24}.$$

Recalling the values of $f(1)$ and $T_3(1)$, we may thus conclude that

$$-\frac{1}{24} \leq \sin 1 - \frac{20}{24} \leq \frac{1}{24} \quad \Longrightarrow \quad \frac{19}{24} \leq \sin 1 \leq \frac{21}{24}.$$

5.5 Trigonometric functions

Theorem 5.51 (Definition of π). The sine function has at least one positive root, and we shall denote by π the smallest such root. Then we have $0 < \pi < 4$, and we also have

$$\sin x > 0 \quad \text{for all } x \in (0, \pi).$$

Theorem 5.52 (Trigonometric formulas). Given any real numbers x and y, one has

$$\sin(x + y) = \sin x \cos y + \sin y \cos x, \qquad \cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Using these two identities, it is now easy to verify that

$$\sin(2x) = 2 \sin x \cos x, \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}, \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

In addition, both $\sin x$ and $\cos x$ are periodic functions in the sense that

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x.$$

Theorem 5.53 (Euler's formula). Letting $i = \sqrt{-1}$ be the imaginary root of -1, one has

$$e^{ix} = \cos x + i \sin x \quad \text{for all } x \in \mathbb{R}.$$

Example 5.54. We use the previous theorem to establish a simple formula for $\sin(3x)$. Here, the key idea is to replace the sine function by an exponential function and then use the main properties of the latter. More precisely, $\sin(3x)$ is the imaginary part of

$$\begin{aligned} e^{3ix} &= (e^{ix})^3 = (\cos x + i \sin x)^3 \\ &= \cos^3 x + 3\cos^2 x(i \sin x) + 3\cos x(i \sin x)^2 + (i \sin x)^3. \end{aligned}$$

Using the fact that $i^2 = -1$ and $i^3 = -i$, we may thus conclude that

$$e^{3ix} = \cos^3 x + 3i \cos^2 x \sin x - 3\cos x \sin^2 x - i \sin^3 x.$$

Since $\sin(3x)$ is given by the imaginary part of e^{3ix}, this also implies that

$$\sin(3x) = 3\cos^2 x \sin x - \sin^3 x.$$

Theorem 5.55 (Polar coordinates). Let (x, y) be a point on the plane other than the origin. Then there exist a unique $r > 0$ and a unique $\theta \in [0, 2\pi)$ such that $x = r\cos\theta$ and $y = r\sin\theta$.

5.6 Area and volume

Definition 5.56 (Area below a graph). Suppose f is continuous and non-negative on $[a, b]$. Then the area that lies between the graph of f and the x-axis is given by the formula

$$\text{Area} = \int_a^b f(x)\, dx.$$

Definition 5.57 (Area between two graphs). Suppose f, g are continuous on $[a, b]$ with

$$f(a) = g(a), \qquad f(b) = g(b), \qquad f(x) \geq g(x) \quad \text{for all } x.$$

Then the area that lies between the graphs of the two functions is given by the formula

$$\text{Area} = \int_a^b \Big[f(x) - g(x) \Big]\, dx.$$

Example 5.58. Let $f(x) = x^2$ and $g(x) = 8\sqrt{x}$. To find the area that lies between the graphs of these two functions, we first note that the graphs intersect when

$$x^2 = 8\sqrt{x} \quad \Longrightarrow \quad x^4 = 64x \quad \Longrightarrow \quad x(x^3 - 64) = 0 \quad \Longrightarrow \quad x = 0, 4.$$

Moreover, a quick sketch shows that the graph of g lies above the graph of f between these two points, so the desired area is given by

$$\text{Area} = \int_0^4 \Big[8\sqrt{x} - x^2 \Big]\, dx = \int_0^4 \Big[8x^{1/2} - x^2 \Big]\, dx = \left[\frac{16x^{3/2}}{3} - \frac{x^3}{3} \right]_0^4 = \frac{64}{3}.$$

Theorem 5.59 (Area of a circle). A circle of radius r has area πr^2.

Definition 5.60 (Volume). Suppose f is continuous on $[a, b]$ and let R be the region that lies between the graph of f and the x-axis. Then the volume generated upon rotation of R around the x-axis is given by the formula

$$\text{Volume} = \int_a^b \pi f(x)^2 \, dx.$$

Example 5.61. We compute the volume of a sphere of radius r. Since such a sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$ around the x-axis, its volume is

$$\text{Volume} = \int_{-r}^{r} \pi f(x)^2 \, dx = \int_{-r}^{r} \pi (r^2 - x^2) \, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4\pi r^3}{3}.$$

Example 5.62. We compute the volume of a cone of radius r and height h. Since such a cone can be obtained by rotating the graph of $f(x) = rx/h$ around the x-axis, its volume is

$$\text{Volume} = \int_0^h \pi f(x)^2 \, dx = \int_0^h \frac{\pi r^2 x^2}{h^2} \, dx = \left[\frac{\pi r^2 x^3}{3h^2} \right]_0^h = \frac{\pi r^2 h}{3}.$$