Chapter 2

Limits and continuity

2.1 The definition of a limit

Definition 2.1 (ε-δ definition). Let f be a function and $y \in \mathbb{R}$ a fixed number. Take x to be a point which approaches y without being equal to y. If there exists a number L that the values $f(x)$ approach as x approaches y, then one expresses this fact by writing

$$\lim_{x \to y} f(x) = L.$$

More precisely, this equation means that given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$0 \neq |x - y| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

If there exists no number L with this property, then we say that $\lim_{x \to y} f(x)$ does not exist.

Proposition 2.2 (Properties of limits). Each of the following statements is true.

(a) The limit of a sum is equal to the sum of the limits, namely

$$\lim_{x \to y} f(x) = L \text{ and } \lim_{x \to y} g(x) = M \quad \Longrightarrow \quad \lim_{x \to y} [f(x) + g(x)] = L + M.$$

(b) The limit of a product is equal to the product of the limits, namely

$$\lim_{x \to y} f(x) = L \text{ and } \lim_{x \to y} g(x) = M \quad \Longrightarrow \quad \lim_{x \to y} [f(x) \cdot g(x)] = LM.$$

(c) When defined, the limit of a quotient is equal to the quotient of the limits, namely

$$\lim_{x \to y} f(x) = L \text{ and } \lim_{x \to y} g(x) = M \neq 0 \quad \Longrightarrow \quad \lim_{x \to y} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Lemma 2.3 (Basic limits). Let $b, y \in \mathbb{R}$ be some fixed numbers.

(i) If $f(x) = b$ for all $x \in \mathbb{R}$, then $\lim_{x \to y} f(x) = b$. In other words, $\lim_{x \to y} b = b$.

(ii) If $f(x) = x$ for all $x \in \mathbb{R}$, then $\lim_{x \to y} f(x) = y$. In other words, $\lim_{x \to y} x = y$.

Theorem 2.4 (Limits of special functions). Let $y \in \mathbb{R}$ be some fixed number.

(a) The limit of a polynomial f can be computed by simple substitution, namely

$$\lim_{x \to y} f(x) = f(y).$$

(b) The limit of a rational function can be computed by simple substitution, namely

$$\lim_{x \to y} \frac{f(x)}{g(x)} = \frac{f(y)}{g(y)}$$

for all polynomials f and g, provided that $g(y) \neq 0$.

2.2 Continuous functions

Definition 2.5 (Continuity). Let f be a function and $y \in \mathbb{R}$ a fixed number. We say that f is continuous at y in the case that

$$\lim_{x \to y} f(x) = f(y).$$

In other words, f is continuous at y if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

We say that f is discontinuous at y, if f is not continuous at y; we say that f is continuous on an interval I, if f is continuous at all points $y \in I$; and we also say that f is continuous, if f is continuous at all points at which it is defined.

Example 2.6 (Discontinuous at one point). Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ll} x & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{array} \right\}.$$

Then f is discontinuous at $y = 1$.

Example 2.7 (Discontinuous at all points). Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Then f is discontinuous at y for all $y \in \mathbb{R}$.

Definition 2.8 (Composition of functions). Given two functions f and g, we define their composition $f \circ g$ by the formula $(f \circ g)(x) = f(g(x))$.

Proposition 2.9 (Continuous functions). Each of the following statements is true.

(a) All polynomials and all rational functions are continuous wherever they are defined.

(b) If each of f, g is continuous at y, then so are their sum $f + g$ and their product fg.

(c) If each of f, g is continuous at y, then so is their quotient f/g, as long as $g(y) \neq 0$.

(d) If g is continuous at y and f is continuous at $g(y)$, then $f \circ g$ is continuous at y.

Definition 2.10 (Open and closed). An interval is said to be open, if it is of the form

$$(-\infty, b), \qquad (a, b), \qquad (a, +\infty).$$

An interval is said to be closed, if it is of the form

$$(-\infty, b], \qquad [a, b], \qquad [a, +\infty).$$

In particular, closed intervals contain their endpoints, whereas open intervals do not.

Lemma 2.11 (Open intervals). Let I be an open interval and let $y \in I$. Then there exists some $\delta > 0$ such that $(y - \delta, y + \delta)$ is a subset of I. Namely, there exists some $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad x \in I.$$

Theorem 2.12 (Functions on open intervals). Suppose the functions f, g agree on an open interval I; that is, suppose $f(x) = g(x)$ for all $x \in I$. If g is continuous on I, then so is f.

Example 2.13 (Checking continuity). Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ll} 2x + 1 & \text{if } x \leq 1 \\ 5 - 2x & \text{if } x > 1 \end{array} \right\}.$$

Then f agrees with a polynomial on the open interval $(-\infty, 1)$, so it is continuous there. It is continuous on $(1, \infty)$ as well for similar reasons. To check continuity at $y = 1$, we note that

$$|f(x) - f(1)| = |f(x) - 3| = \left\{ \begin{array}{ll} |2x - 2| & \text{if } x \leq 1 \\ |2 - 2x| & \text{if } x > 1 \end{array} \right\} = |2x - 2|.$$

Given any $\varepsilon > 0$, we can then set $\delta = \varepsilon/2$ to find that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - f(1)| = 2 \cdot |x - 1| < 2\delta = \varepsilon.$$

This establishes continuity at $y = 1$ as well, so f is continuous at all points.

2.3 Properties of continuity

Lemma 2.14 (Continuity and positivity). Suppose that f is continuous at y.

(a) If $f(y) > 0$, then there exists some $\delta > 0$ such that $f(x) > 0$ for all $x \in (y-\delta, y+\delta)$.

(b) If $f(y) < 0$, then there exists some $\delta > 0$ such that $f(x) < 0$ for all $x \in (y-\delta, y+\delta)$.

Theorem 2.15 (BOLZANO'S THEOREM). Suppose that f is continuous on $[a, b]$.

(a) If $f(a) < 0 < f(b)$, then there exists some $x \in (a, b)$ such that $f(x) = 0$.

(b) If $f(b) < 0 < f(a)$, then there exists some $x \in (a, b)$ such that $f(x) = 0$.

Application 2.16 (Existence of roots). Let f be the function defined by $f(x) = x^3 - x - 1$. Being a polynomial, f is then continuous on $[1, 2]$. Since $f(1) = -1 < 0$ and $f(2) = 5 > 0$, we can then apply Bolzano's theorem to find some $x \in (1, 2)$ such that $f(x) = 0$.

Theorem 2.17 (Square roots). Given any $y \geq 0$, there exists a unique real number $x \geq 0$ such that $x^2 = y$. We shall denote this particular number by $x = \sqrt{y}$.

Proposition 2.18 (Continuous functions). Each of the following statements is true.

(a) All polynomials and all rational functions are continuous wherever they are defined.

(b) Sums, products, quotients and compositions of continuous functions are continuous.

(c) The square root function, which is defined by $f(x) = \sqrt{x}$ for all $x \geq 0$, is continuous.

(d) The absolute value function, which is defined by $f(x) = |x|$ for all $x \in \mathbb{R}$, is continuous.

Example 2.19 (Limits by simple substitution). Using the proposition above, we find

$$\lim_{x \to 2} \sqrt{x^2 + 5} = \sqrt{2^2 + 5} = \sqrt{9} = 3$$

because $\sqrt{x^2 + 5}$ is the composition of continuous functions. For similar reasons, one also has

$$\lim_{x \to 4} |\sqrt{x} - x| = |\sqrt{4} - 4| = |2 - 4| = 2.$$

Theorem 2.20 (Quadratic formula). Let $a, b, c \in \mathbb{R}$ be fixed real numbers with $a \neq 0$.

(a) If $b^2 - 4ac \geq 0$, then the quadratic equation $ax^2 + bx + c = 0$ has roots

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

(b) If $b^2 - 4ac < 0$, then the quadratic equation $ax^2 + bx + c = 0$ has no roots.

Example 2.21. To solve the inequality $x^2 - 3x + 2 < 0$, one solves the equality $x^2 - 3x + 2 = 0$ first. Since the two roots are $x_1 = 1$ and $x_2 = 2$, we get $x^2 - 3x + 2 = (x - 1)(x - 2)$. Then the table below suggests that $x^2 - 3x + 2 < 0$ if and only if $1 < x < 2$.

x		1		2	
$x - 1$	$-$		$+$		$+$
$x - 2$	$-$		$-$		$+$
$x^2 - 3x + 2$	$+$		$-$		$+$

Theorem 2.22 (INTERMEDIATE VALUE THEOREM). Suppose that f is continuous on a closed interval $[a, b]$. Then f attains all values between $f(a)$ and $f(b)$. More precisely,

(a) given any $f(a) < c < f(b)$, there exists some $x \in (a, b)$ such that $f(x) = c$.

(b) given any $f(b) < c < f(a)$, there exists some $x \in (a, b)$ such that $f(x) = c$.

Theorem 2.23 (Continuity and lower/upper bounds). If f is continuous on a closed interval $[a, b]$, then its values $f(x)$ have both a lower bound and an upper bound. That is, there exist numbers $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leq f(x) \leq M_2$ for all $x \in [a, b]$.

Remark. This theorem is not generally valid for other kinds of intervals. For instance, it is easy to check that $f(x) = 1/x$ has no upper bound on $(0, 1)$.

Theorem 2.24 (EXTREME VALUE THEOREM). Suppose f is continuous on a closed interval $[a, b]$. Then f attains both its minimum and its maximum value on $[a, b]$. That is, there exist points $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

Remark. This theorem is not generally valid for other kinds of intervals. For instance, it should be clear that $f(x) = x$ attains neither a minimum nor a maximum value on $(0, 1)$.

2.4 Limits at infinity

Definition 2.25 (ε-N definition). Let f be a function. If there exists a number L that the values $f(x)$ approach for large enough values of x, then one expresses this fact by writing

$$\lim_{x \to +\infty} f(x) = L.$$

More precisely, this equation means that given any $\varepsilon > 0$, there exists some $N > 0$ such that

$$x > N \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

If there exists no number L with this property, then we say that $\lim_{x \to +\infty} f(x)$ does not exist.

Example 2.26. Given any natural number n, one has $\lim_{x \to +\infty} \frac{1}{x^n} = 0$.

Proposition 2.27 (Properties of limits). Each of the following statements is true.

(a) The limit of a sum is equal to the sum of the limits, namely

$$\lim_{x\to+\infty} f(x) = L \text{ and } \lim_{x\to+\infty} g(x) = M \quad\Longrightarrow\quad \lim_{x\to+\infty} [f(x)+g(x)] = L+M.$$

(b) The limit of a product is equal to the product of the limits, namely

$$\lim_{x\to+\infty} f(x) = L \text{ and } \lim_{x\to+\infty} g(x) = M \quad\Longrightarrow\quad \lim_{x\to+\infty} [f(x)\cdot g(x)] = LM.$$

(c) When defined, the limit of a quotient is equal to the quotient of the limits, namely

$$\lim_{x\to+\infty} f(x) = L \text{ and } \lim_{x\to+\infty} g(x) = M \neq 0 \quad\Longrightarrow\quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Example 2.28 (Limits of rational functions at infinity). Given a rational function, one can easily compute its limit as $x \to +\infty$. The main step is to divide both the numerator and the denominator by the highest power of x that appears downstairs. For instance, one has

$$\lim_{x\to+\infty} \frac{2x^3+3x^2+5}{x^3-2x^2+x} = \lim_{x\to+\infty} \frac{2+\frac{3}{x}+\frac{5}{x^3}}{1-\frac{2}{x}+\frac{1}{x^2}} = \frac{2+0+0}{1-0+0} = 2$$

by Example 2.26 and since $x \neq 0$ here. Using the same argument, one similarly finds that

$$\lim_{x\to+\infty} \frac{x^2+4x-3}{x^3-7x+9} = \lim_{x\to+\infty} \frac{\frac{1}{x}+\frac{4}{x^2}-\frac{3}{x^3}}{1-\frac{7}{x^2}+\frac{9}{x^3}} = \frac{0+0-0}{1-0+0} = 0.$$

Definition 2.29 (Limits at $-\infty$). The limit of a function as $x \to -\infty$ is defined by

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(-x),$$

provided that this limit exists. We say that $\lim_{x\to-\infty} f(x)$ does not exist, otherwise.

Remark. In view of the definition above, properties for limits as $x \to -\infty$ follow from the corresponding properties for limits as $x \to +\infty$. In particular, one also has

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

for each $n \in \mathbb{N}$, and the limit of a sum/product/quotient is equal to the sum/product/quotient of the limits, respectively. I shall not bother to list these facts in a separate proposition. Using these facts as above, one can then compute limits of rational functions such as

$$\lim_{x\to-\infty} \frac{3x^2-3x+4}{x^2-4x+6} = \lim_{x\to-\infty} \frac{3-\frac{3}{x}+\frac{4}{x^2}}{1-\frac{4}{x}+\frac{6}{x^2}} = \frac{3-0+0}{1-0+0} = 3,$$

for instance. Some other methods for computing limits will be given in the next chapter.