Chapter 1

Some basic concepts

1.1 The set of real numbers

Axioms 1.1 (Axioms for addition). Each of the following statements is true.

(A1) **Commutative law**: we have $x + y = y + x$ for all $x, y \in \mathbb{R}$.

(A2) **Associative law**: we have $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$.

(A3) **Zero**: There exists an element $0 \in \mathbb{R}$ such that $0 + x = x$ for all $x \in \mathbb{R}$.

(A4) **Negatives**: Given any $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ such that $x + y = 0$. We shall denote this unique element by $y = -x$ and also write $z - w$ instead of $z + (-w)$.

Lemma 1.2 (Addition in $\mathbb{R}$). The following rules hold for addition in $\mathbb{R}$.

(a) **Cancellation law**: we have $x + y = x + z \implies y = z$ for all $x, y, z \in \mathbb{R}$.

(b) **Two negatives cancel**: we have $-(-x) = x$ for all $x \in \mathbb{R}$.

Axioms 1.3 (Axioms for multiplication). Each of the following statements is true.

(M1) **Commutative law**: we have $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$.

(M2) **Associative law**: we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$.

(M3) **One**: There exists a real number $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$.

(M4) **Inverses**: Given any real number $x \neq 0$, there exists a unique $y \in \mathbb{R}$ such that $x \cdot y = 1$. We shall denote this unique element by $y = 1/x$ and also write z/w instead of $z \cdot (1/w)$.

(M5) **Distributive law**: we have $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.

Lemma 1.4 (Multiplication in $\mathbb{R}$). The following rules hold for multiplication in $\mathbb{R}$.

(a) We have $0 \cdot x = 0$ for all $x \in \mathbb{R}$.

(b) We have $x \cdot y = 0 \implies x = 0$ or $y = 0$.

(c) We have $(-x)y = -(xy) = x(-y)$ for all $x, y \in \mathbb{R}$.

(d) **Two negatives cancel**: we have $(-x)(-y) = xy$ for all $x, y \in \mathbb{R}$.

(e) **Two inverses cancel**: given any $x \neq 0$, we have $1/(1/x) = x$.

(f) **Cancellation law**: if $xy = xz$ and $x \neq 0$, then we must have $y = z$.

Definition 1.5 (Ordered set). A set S is said to be ordered, if there is a relation $<$ which is defined between the elements of S in such a way that the following properties hold.

(O1) **Trichotomy**: given any two elements $x, y \in S$, exactly one of the three statements

$$x < y, \qquad x = y, \qquad y < x$$

is true. We shall write $x \leq y$ whenever either of the first two statements is true.

(O2) **Transitivity**: if $x < y$ and $y < z$, then $x < z$.

Axioms 1.6 (Axioms for inequalities). The set $\mathbb{R}$ of all real numbers is an ordered set and each of the following statements is true.

(I1) **Cancellation law**: we have $x + y < x + z \iff y < z$ for all $x, y, z \in \mathbb{R}$.

(I2) **Products of positive numbers**: if $x > 0$ and $y > 0$, then $xy > 0$.

Notation. We shall use the notation $2 = 1 + 1$, $3 = 1 + 1 + 1$ and so on for the real numbers obtained by adding 1 to itself. We shall similarly write $x^2 = x \cdot x$, $x^3 = x \cdot x \cdot x$ and so on.

Lemma 1.7 (Inequalities in $\mathbb{R}$). The following rules hold for inequalities in $\mathbb{R}$.

(a) We have $x > 0 \iff -x < 0$ for all $x \in \mathbb{R}$.

(b) **Squares are non-negative**: we have $x^2 \geq 0$ for all $x \in \mathbb{R}$.

(c) **Multiplying by positive numbers**: if $x > y$ and $z > 0$, then $xz > yz$.

(d) **Multiplying by negative numbers**: if $x > y$ and $z < 0$, then $xz < yz$.

(e) **Inverting positive numbers**: if $x > y$ are both positive, then $\frac{1}{x} < \frac{1}{y}$.

(f) **Adding inequalities**: if $x > y$ and $z > w$, then $x + z > y + w$.

Lemma 1.8 (Useful identities). Given any real numbers x and y, we have the identities

$$(x+y)^2 = x^2 + 2xy + y^2, \qquad x^2 - y^2 = (x-y)(x+y).$$

Application 1.9 (Completing squares). Using the first identity above, one finds that

$$x^2 - 4x = x^2 - 4x + 4 - 4 = (x-2)^2 - 4 \geq -4$$

with equality if and only if $x = 2$. Using a similar computation, one finds that

$$-2x^2 + 4x = -2(x^2 - 2x + 1 - 1) = -2(x-1)^2 + 2 \leq 2$$

with equality if and only if $x = 1$.

Definition 1.10 (Absolute value). The absolute value of a real number x is defined by

$$|x| = \left\{ \begin{array}{cl} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{array} \right\}.$$

As you can easily convince yourselves, $|x-y|$ measures the distance between x and y.

Lemma 1.11 (Properties of absolute values). Each of the following statements is true.

(a) We have $|x| \geq 0$ and also $|x| \geq x$ for all $x \in \mathbb{R}$.

(b) We have $|x|^2 = x^2$ for all $x \in \mathbb{R}$ and also $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.

(c) Given any $\varepsilon > 0$, we have $|x| < \varepsilon \iff -\varepsilon < x < \varepsilon$.

(d) **Removing squares**: If x, y are both non-negative, then $x^2 \leq y^2 \iff x \leq y$.

(e) **Triangle inequality**: We have $|x+y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

1.2 Upper and lower bounds

Notation. We say that A is a subset of B and we write $A \subset B$ whenever every element of A is an element of B as well. A set that has no elements is said to be empty.

Definition 1.12 (max and sup). Suppose that A is a nonempty subset of $\mathbb{R}$.

(a) The largest element of A, should one exist, is called the maximum of A.

(b) If there exists some $x \in \mathbb{R}$ such that $x \geq a$ for all $a \in A$, we say that x is an upper bound of A and we also say that A is bounded from above.

(c) The least upper bound of A, should one exist, is called the supremum of A.

The maximum and the supremum of A are denoted by $\max A$ and $\sup A$, respectively. Note that $\max A$ is necessarily an element of A, whereas $\sup A$ need not be.

Warning. By definition, the empty set has neither a maximum nor a supremum.

Example 1.13. Let $A = \{x \in \mathbb{R} : x \leq 0\}$. Then $\max A = 0$ and $\sup A = 0$.

Example 1.14. Let $A = \{x \in \mathbb{R} : x < 0\}$. Then $\sup A = 0$, still $\max A$ does not exist.

Example 1.15. Let $A = \{x \in \mathbb{R} : x > 0\}$. Then neither $\sup A$ nor $\max A$ exists.

Theorem 1.16 (Average). Given any two real numbers $x < y$, there exists a real number z such that $x < z < y$. In fact, the average $z = \frac{x+y}{2}$ of the two numbers is always such.

Axiom 1.17 (Completeness). Every nonempty subset of $\mathbb{R}$ that has an upper bound must necessarily have a least upper bound, namely a supremum.

Notation. We shall denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of all natural numbers, and we shall similarly denote by $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ the set of all integers.

Lemma 1.18. The set $\mathbb{N}$ of all natural numbers has no upper bound.

Theorem 1.19 (Large integers). Given any $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $n > x$.

Example 1.20. Let $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. Then $\max A$ does not exist, while $\sup A = 1$.

Definition 1.21 (min and inf). Suppose that A is a nonempty subset of $\mathbb{R}$.

(a) The smallest element of A, should one exist, is called the minimum of A.

(b) If there exists some $x \in \mathbb{R}$ such that $x \leq a$ for all $a \in A$, we say that x is a lower bound of A and we also say that A is bounded from below.

(c) The greatest lower bound of A, should one exist, is called the infimum of A.

The minimum and the infimum of A are denoted by $\min A$ and $\inf A$, respectively. Note that $\min A$ is necessarily an element of A, whereas $\inf A$ need not be.

Warning. By definition, the empty set has neither a minimum nor an infimum.

Theorem 1.22 (Completeness). Every nonempty subset of $\mathbb{R}$ that has a lower bound must necessarily have a greatest lower bound, namely an infimum.

Theorem 1.23 (min$\rightsquigarrow$inf and max$\rightsquigarrow$sup). Let A be a nonempty subset of $\mathbb{R}$.

(a) If $\min A$ exists, then $\inf A$ exists and the two are equal.

(b) If $\max A$ exists, then $\sup A$ exists and the two are equal.

Theorem 1.24 (inf$\rightsquigarrow$min and sup$\rightsquigarrow$max). Let A be a nonempty subset of $\mathbb{R}$.

(a) If $\inf A$ exists and $\inf A \in A$, then $\min A$ exists and the two are equal.

(b) If $\sup A$ exists and $\sup A \in A$, then $\max A$ exists and the two are equal.

Example 1.25. Let $A = \{x \in \mathbb{R} : 0 \le x < 1\}$. Then $\max A$ does not exist, while

$$\sup A = 1, \qquad \min A = \inf A = 0.$$

Example 1.26. Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\min A$ does not exist, while

$$\inf A = 0, \qquad \max A = \sup A = 1.$$

Theorem 1.27 (inf/sup of subsets). Suppose that $A \subset B$ are nonempty subsets of $\mathbb{R}$.

(a) If $\inf B$ exists, then $\inf A$ exists and we have $\inf A \ge \inf B$.

(b) If $\sup B$ exists, then $\sup A$ exists and we have $\sup A \le \sup B$.

Loosely speaking, this theorem says that larger sets have larger suprema but smaller infima.

Theorem 1.28 (Subsets of $\mathbb{Z}$). Suppose that A is a nonempty subset of $\mathbb{Z}$.

(a) If A has a lower bound, then A must actually have a minimum.

(b) If A has an upper bound, then A must actually have a maximum.

Example 1.29. Let $A = \{x \in \mathbb{Z} : x < 1\}$. Then $\max A = 0$, still $\min A$ does not exist.

Theorem 1.30 (Induction). To show that some statement holds for <u>all</u> $n \in \mathbb{N}$, it suffices to

① check that the statement holds when $n = 1$;

② assume that the statement holds for <u>some</u> $n \in \mathbb{N}$ and show that it holds for $n + 1$.

Example 1.31. We use induction to establish the formula

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

① To show that the formula holds when $n = 1$, we have to check that

$$1 = \frac{1 \cdot (1+1)}{2} \qquad \Longleftrightarrow \qquad 2 = 1 + 1,$$

and this is certainly true. In particular, the given formula does hold when $n = 1$.
② Suppose now that the formula holds for some $n \in \mathbb{N}$, namely suppose that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Adding $n + 1$ to both sides and simplifying, we then get

$$1 + 2 + \ldots + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

This shows that the formula holds for $n + 1$ as well, so it actually holds for all $n \in \mathbb{N}$.

Definition 1.32 (Rationals). The set $\mathbb{Q}$ of all rationals is defined by

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}.$$

As one can easily check, both the sum and the product of two rationals is a rational itself. The same is true for the inverse of nonzero rationals, hence also for quotients of rationals. A real number which is not rational is called irrational.

Theorem 1.33 (Rationals in between). Given any two real numbers $x < y$, there exists a rational number $z \in \mathbb{Q}$ such that $x < z < y$.

Theorem 1.34 (Square root of 2). There exists a unique, positive real number y such that

$$y^2 = 2.$$

Moreover, this real number y is not rational, and we shall denote it by $y = \sqrt{2}$.

Theorem 1.35 (Irrationals in between). Given any two real numbers $x < y$, there exists an irrational number z such that $x < z < y$.

1.3 Functions

Notation (Intervals). Given any real numbers a, b with $a < b$, we shall denote by

$$(a, b), \qquad [a, b), \qquad (a, b], \qquad [a, b]$$

the sets of all real numbers that lie between a and b. In each case, a square bracket is assigned to points which belong to the set and a regular bracket to those which do not. For instance,

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

and so on. We shall also use the symbols $-\infty$ and $+\infty$ to denote intervals such as

$$(-\infty, b), \qquad (-\infty, b], \qquad (a, +\infty), \qquad [a, +\infty).$$

In this case, the intervals are defined by

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

and so on. The symbols $-\infty$ and $+\infty$ are not elements of any interval themselves because they are not real numbers; they do not even satisfy the usual rules of arithmetic such as $0 \cdot \infty = 0$ and $\infty/\infty = 1$, for instance. These symbols are merely introduced to simplify the notation.

Definition 1.36 (Functions). A function f is a rule or formula which is defined for some real numbers x and assigns a particular value $f(x)$ to each admissible value of x. The set of all real numbers x for which f is defined is called the domain of f.

Definition 1.37 (Sums, products and quotients). Given two functions f and g, we define their sum $f+g$, product $f \cdot g$ and quotient f/g in the obvious way. For instance,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

and so on. Note that the quotient f/g is not defined at points $x \in \mathbb{R}$ at which $g(x) = 0$.

Notation (inf/sup of functions). Given a function f and a nonempty subset A of the real numbers, we shall write

$$\inf_{x \in A} f(x) \quad \text{instead of} \quad \inf\{f(x) : x \in A\},$$

and we shall also write

$$\inf_{0<x<1} f(x) \quad \text{instead of} \quad \inf\{f(x) : 0 < x < 1\}.$$

Moreover, we shall use a similar notation for the supremum of a function. In the case that A is the set of all real numbers, one typically omits the subscript $x \in A$, thus writing

$$\inf f(x) \quad \text{instead of} \quad \inf_{x \in \mathbb{R}} f(x) = \inf\{f(x) : x \in \mathbb{R}\}.$$

Warning. Define $f(x) = x$ and $g(x) = -x$ for all $x \in [0,1]$. Then we have

$$\begin{aligned} \inf_{0 \le x \le 1} f(x) &= \inf\{x : 0 \le x \le 1\} = 0, \\ \inf_{0 \le x \le 1} g(x) &= \inf\{-x : 0 \le x \le 1\} = -1. \end{aligned}$$

In particular, the sum of the infima is not equal to the infimum of the sum

$$\inf_{0 \le x \le 1} [f(x) + g(x)] = \inf\{x - x : 0 \le x \le 1\} = 0.$$

Theorem 1.38 (inf/sup of sums). Given any two functions f and g, one has

$$\inf_{x \in A} [f(x) + g(x)] \ge \inf_{x \in A} f(x) + \inf_{x \in A} g(x)$$

and also

$$\sup_{x \in A} [f(x) + g(x)] \le \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$$

for all sets $A \subset \mathbb{R}$ for which the infima/suprema on the right hand side exist.

Theorem 1.39 (inf/sup and inequalities). Loosely speaking, applying either inf or sup to both sides of an inequality preserves the inequality. More formally, one has

$$f(x) \le g(x) \quad \text{for all } x \in A \quad \Longrightarrow \quad \inf_{x \in A} f(x) \le \inf_{x \in A} g(x)$$

and similarly

$$f(x) \le g(x) \quad \text{for all } x \in A \quad \Longrightarrow \quad \sup_{x \in A} f(x) \le \sup_{x \in A} g(x)$$

for all functions f, g and all sets $A \subset \mathbb{R}$ for which the above infima/suprema exist.

Definition 1.40 (Special kinds of functions). A function f is said to be:

- constant, if there exists some $b \in \mathbb{R}$ such that $f(x) = b$ for all $x \in \mathbb{R}$.
- linear, if there exist some $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.
- a polynomial, if there exist real numbers $a_0, a_1, \ldots, a_n$ such that

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \qquad \text{for all } \ x \in \mathbb{R}.$$

- a rational function, if there exist polynomials P, Q such that

$$f(x) = \frac{P(x)}{Q(x)} \qquad \text{for all } \ x \in \mathbb{R} \ \text{ with } \ Q(x) \neq 0.$$

Example 1.41 (Division of polynomials). Using the division algorithm, one finds that

$$\frac{x^3 + 2x^2 - 3}{x - 1} = x^2 + 3x + 3 \quad \text{for all } x \neq 1.$$

Using a similar computation, one also finds that

$$\frac{x^5 - x^3 + x^2 - 2x + 1}{x^2 + 1} = x^3 - 2x + 1 \quad \text{for all } x \in \mathbb{R}.$$