MA121, 2006 Exam #1
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \left\{n \in \mathbb{N} : \frac{1}{n} < \frac{2}{3}\right\}$

(b) $B = \{x \in \mathbb{R} : x^2 < -1\}$

(c) $C = \{x \in \mathbb{Z} : x \geq 3 \text{ and } 2x < 7\}$

(d) $D = \{x \in \mathbb{R} : |x+1| < 1\}$

• A complete list of answers is provided by the following table.

	min	inf	max	sup
A	2	2	DNE	DNE
B	DNE	DNE	DNE	DNE
C	3	3	3	3
D	DNE	-2	DNE	0

• The set A contains all $n \in \mathbb{N}$ with $n > \frac{3}{2}$; this means that $A = \{2, 3, 4, \ldots\}$.

• The set B is empty because $x^2 \geq 0 > -1$ for each $x \in \mathbb{R}$.

• The set C contains all integers x with $3 \leq x < \frac{7}{2}$; this means that $C = \{3\}$.

• The set D contains the real numbers x whose distance from -1 is strictly less than 1. Based on this fact, it is easy to see that $D = (-2, 0)$.

2. *Show that* $2^n \geq n + 1$ *for all* $n \in \mathbb{N}$.

• We use induction to prove the given inequality for all $n \in \mathbb{N}$.

• When $n = 1$, the given inequality holds with equality because $2^1 = 2 = 1 + 1$.

• Suppose that the inequality holds for some n, in which case

$$2^n \geq n + 1.$$

Multiplying this inequality with the positive number 2, we then get

$$2^{n+1} \geq 2(n+1) = 2n + 2 \geq n + 2 = (n+1) + 1$$

because $n \geq 0$ for all $n \in \mathbb{N}$. This proves the given inequality for $n + 1$, as needed.

3. *Show that there exists some* $0 < x < 1$ *such that* $2x^2 + 3x^3 = x^5 + 1$.

Let $f(x) = 2x^2 + 3x^3 - x^5 - 1$ for all $x \in [0, 1]$. Being a polynomial, f is then continuous on the closed interval $[0, 1]$. Once we now note that

$$f(0) = -1 < 0, \qquad f(1) = 2 + 3 - 1 - 1 = 3 > 0,$$

we may use Bolzano's theorem to conclude that $f(x) = 0$ for some $x \in (0, 1)$. This also implies that $2x^2 + 3x^3 = x^5 + 1$ for some $0 < x < 1$, as needed.

4. *Let f be a function such that $|f(x)-1| \leq 2|x|$ for all $x \in \mathbb{R}$. Show that* $\lim_{x\to 0} f(x) = 1$.

Let $\varepsilon > 0$ be given and set $\delta = \varepsilon/2$. Then $\delta > 0$ and we easily find that

$$0 \neq |x-0| < \delta \quad \Longrightarrow \quad |f(x)-1| \leq 2|x| < 2\delta = \varepsilon.$$

5. *Let A, B be nonempty subsets of $\mathbb{R}$ such that* $\sup A \leq b$ *for all $b \in B$. Show that*

$$\inf A \leq \inf B.$$

As a hint, you might wish to show that $\inf A \leq \sup A$ *and that* $\sup A \leq \inf B$*, instead.*

- Since $\inf A$ is a lower bound of A and $\sup A$ is an upper bound of A, one has

$$\inf A \leq a \leq \sup A$$

for all $a \in A$. This certainly implies that $\inf A \leq \sup A$.

- Since $\sup A \leq b$ for all $b \in B$ by assumption, $\sup A$ is a lower bound of B, so it can only be as large as the greatest lower bound of B. This shows that $\sup A \leq \inf B$.

- In view of these observations, one now finds that $\inf A \leq \sup A \leq \inf B$, as needed.

6. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} \frac{x^3+x^2-2}{x-1} & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{array} \right\}.$$

Show that f is continuous at all points. As a hint, one may avoid the ε-δ definition here.

- Assuming that $x \neq 1$, one may use division of polynomials to write

$$f(x) = \frac{x^3+x^2-2}{x-1} = x^2+2x+2.$$

This means that f agrees with a polynomial on the open intervals $(-\infty, 1)$ and $(1, +\infty)$. Since all polynomials are continuous, f itself must be continuous on these intervals.

- To check continuity at the remaining point $y = 1$, we have to show that

$$\lim_{x\to 1} f(x) = f(1).$$

Let us then try to compute this limit. Assuming that $x \neq 1$, as we may, we get

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3+x^2-2}{x-1} = \lim_{x\to 1} (x^2+2x+2).$$

Since limits of polynomials can be computed by simple substitution, this also implies

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2+2x+2) = 1^2+2+2 = 5 = f(1).$$

In particular, f is continuous at $y = 1$ as well, so f is continuous at all points.

7. *Given the set* $A = \left\{\frac{x}{x^2+1} : x \in \mathbb{R}\right\}$, *show that* $\inf A = -\frac{1}{2}$.

Since both x^2+1 and 2 are positive numbers, it is easy to check that

$$\begin{aligned} \frac{x}{x^2+1} \geq -\frac{1}{2} \quad &\iff \quad 2x \geq -x^2 - 1 \quad \iff \quad x^2 + 2x + 1 \geq 0 \\ &\iff \quad (x+1)^2 \geq 0. \end{aligned}$$

Note that the last inequality obviously holds and that we do have equality when $x = -1$. Thus, the first inequality holds as well. This makes $-\frac{1}{2}$ an element of A which is at least as small as any other element of A, so $\min A = -\frac{1}{2}$. Since a minimum exists in this case, an infimum also does and the two are equal; so $\inf A = -\frac{1}{2}$ as well.

8. *Show that the function f defined by*

$$f(x) = \left\{ \begin{array}{cl} 2x & \text{if } x \leq 1 \\ x+3 & \text{if } x > 1 \end{array} \right\}$$

is discontinuous at $y = 1$.

We will show that the ε-δ definition of continuity fails when $\varepsilon = 2$. Suppose it does not fail. Since $f(1) = 2$, there must then exist some $\delta > 0$ such that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - 2| < 2. \tag{$*$}$$

Let us now examine the last equation for the choice $x = 1 + \frac{\delta}{2}$. On one hand, we have

$$|x - 1| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x) - 2| = |x + 3 - 2| = x + 1 = 2 + \frac{\delta}{2} > 2$$

because $x = 1 + \frac{\delta}{2} > 1$ here. This actually violates the conclusion in equation $(*)$.