MA2E01 Tutorial solutions #9

1. Find the flux of $\boldsymbol{F}(x, y, z) = \langle x, y, 2z\rangle$ through the surface σ when σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 0$ and $z = 1$, oriented upwards.

In this case, we have $z = f(x, y) = \sqrt{x^2 + y^2}$, so

$$\boldsymbol{n}\, dS = \langle -f_x, -f_y, 1\rangle \; dx\, dy = \left\langle -\frac{2x}{2\sqrt{x^2 + y^2}}, -\frac{2y}{2\sqrt{x^2 + y^2}}, 1 \right\rangle \; dx\, dy.$$

Taking the dot product with $\boldsymbol{F} = \left\langle x, y, 2\sqrt{x^2 + y^2}\right\rangle$, we now find that

$$\begin{aligned}
\text{Flux} = \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iint_\sigma \left(-\frac{x^2}{\sqrt{x^2 + y^2}} - \frac{y^2}{\sqrt{x^2 + y^2}} + 2\sqrt{x^2 + y^2} \right) dx\, dy \\
&= \iint_\sigma \sqrt{x^2 + y^2}\, dx\, dy \\
&= \int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta = \int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{2\pi}{3}.
\end{aligned}$$

2. Find the flux of $\boldsymbol{F}(x, y, z) = \langle z, 4x, y\rangle$ through the surface σ when σ is the part of the plane $2x + y + z = 4$ that lies in the first octant, oriented upwards.

Arguing as before, we get $z = f(x, y) = 4 - 2x - y$, hence also

$$\boldsymbol{n}\, dS = \langle -f_x, -f_y, 1\rangle \; dx\, dy = \langle 2, 1, 1\rangle \; dx\, dy.$$

Since $\boldsymbol{F} = \langle z, 4x, y\rangle = \langle 4 - 2x - y, 4x, y\rangle$, this actually implies that

$$\begin{aligned}
\text{Flux} = \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iint_\sigma (8 - 4x - 2y + 4x + y)\, dx\, dy \\
&= \int_0^4 \int_0^{(4-y)/2} (8 - y)\, dx\, dy \\
&= \int_0^4 \frac{(4 - y)(8 - y)}{2}\, dy = \frac{1}{2} \int_0^4 (32 - 12y + y^2)\, dy \\
&= \frac{1}{2} \left(32 \cdot 4 - 6 \cdot 4^2 + \frac{4^3}{3} \right) = \frac{80}{3}.
\end{aligned}$$

3. Use the divergence theorem to find the outward flux of $\boldsymbol{F}(x, y, z) = \langle x^3, y^3, z^3\rangle$ through the (entire) surface of the cylinder $y^2 + z^2 = 4$ that lies between $x = 0$ and $x = 2$.

In this case, $\operatorname{div} \boldsymbol{F} = 3x^2 + 3y^2 + 3z^2 = 3x^2 + 12$, so the divergence theorem gives

$$\text{Outward flux} = \iiint_G (3x^2 + 12)\, dx\, dy\, dz.$$

Writing the equation $y^2 + z^2 = 4$ in terms of polar coordinates, we conclude that

$$
\begin{aligned}
\text{Outward flux} &= \int_0^{2\pi} \int_0^2 \int_0^2 r(3x^2 + 12)\, dx\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \Big[r(x^3 + 12x) \Big]_{x=0}^2 dr\, d\theta = \int_0^{2\pi} \int_0^2 32r\, dr\, d\theta \\
&= \int_0^{2\pi} \Big[16r^2 \Big]_{r=0}^2 d\theta = 16 \cdot 2^2 \cdot 2\pi = 128\pi.
\end{aligned}
$$

4. Use Stokes' theorem to compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ when $\boldsymbol{F}(x, y, z) = \langle z^2, y^2, x\rangle$ and C is the rectangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 2, 2)$ and $(0, 2, 2)$, oriented counterclockwise as one looks down to the plane $y = z$ that contains the rectangle.

A short computation gives $\operatorname{curl} \boldsymbol{F} = \langle 0, 2z - 1, 0\rangle$ and we also have

$$
z = f(x, y) = y \quad \Longrightarrow \quad \boldsymbol{n}\, dS = \langle -f_x, -f_y, 1\rangle\ dx\, dy = \langle 0, -1, 1\rangle\ dx\, dy.
$$

According to Stokes' theorem then, the given integral is equal to

$$
\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS &= \int_0^2 \int_0^1 (1 - 2z)\, dx\, dy \\
&= \int_0^2 \int_0^1 (1 - 2y)\, dx\, dy \\
&= \int_0^2 (1 - 2y)\, dy = 2 - 2^2 = -2.
\end{aligned}
$$