MA2E01 Tutorial solutions #8

1. Let R be the region bounded by the curves $y = x$ and $y = x^2$. Let C be the boundary of this region oriented counterclockwise. Use Green's theorem to evaluate

$$\oint_C 2xy\,dx + (x^2 + 2xy)\,dy.$$

According to Green's theorem, the given integral is equal to

$$\begin{aligned}\iint_R \Big[(x^2+2xy)_x - (2xy)_y\Big]\,dA &= \iint_R \Big[2x+2y-2x\Big]\,dA \\ &= \int_0^1\int_{x^2}^{x} 2y\,dy\,dx \\ &= \int_0^1 (x^2-x^4)\,dx = \frac{1}{3}-\frac{1}{5} = \frac{2}{15}.\end{aligned}$$

2. Use Green's theorem to find the work done by $\boldsymbol{F} = \langle 2xy, x^2+2xy\rangle$ while moving a particle from $(2,0)$ to $(-2,0)$ along the upper semicircle $x^2+y^2=4$ and then back to the point $(2,0)$ along the x-axis.

Since the vector field $\boldsymbol{F}$ is the same as in the previous problem, we have

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R 2y\,dA &= \int_0^\pi\int_0^2 2r^2\sin\theta\,dr\,d\theta \\ &= \int_0^\pi \frac{2\cdot 2^3}{3}\sin\theta\,d\theta = \frac{16}{3}(-\cos\pi+\cos 0) = \frac{32}{3}.\end{aligned}$$

3. Compute the surface integral $\iint_\sigma z^2\,dS$ when σ is the part of the cylinder $y^2+z^2=4$ that lies between the planes $x=0$ and $x=3$.

First, we note that a parametric equation of the given cylinder is

$$\boldsymbol{r} = \langle x,y,z\rangle = \langle x, 2\cos\theta, 2\sin\theta\rangle$$

with $0\leq\theta\leq 2\pi$ and $0\leq x\leq 3$. Using this equation, we now get

$$\boldsymbol{r}_x\times\boldsymbol{r}_\theta = \langle 1,0,0\rangle\times\langle 0,-2\sin\theta,2\cos\theta\rangle = \langle 0,-2\cos\theta,-2\sin\theta\rangle$$

and then the given integral becomes

$$\begin{aligned}\iint_\sigma z^2\,dS &= \int_0^{2\pi}\int_0^3 4\sin^2\theta\cdot||\boldsymbol{r}_x\times\boldsymbol{r}_\theta||\,dx\,d\theta \\ &= \int_0^{2\pi}\int_0^3 8\sin^2\theta\,dx\,d\theta = \int_0^{2\pi} 24\sin^2\theta\,d\theta \\ &= \int_0^{2\pi} 12\Big(1-\cos(2\theta)\Big)\,d\theta = \Big[12\theta - 6\sin(2\theta)\Big]_0^{2\pi} = 24\pi.\end{aligned}$$

4. Find the mass of the lamina that has constant density δ and occupies the part of the plane $x + y + z = 1$ which lies in the first octant.

In this case, we can express z as a function of x and y, so we get

$$z = f(x, y) = 1 - x - y \quad \Longrightarrow \quad dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy = \sqrt{3}\, dx\, dy.$$

The mass of the lamina is the integral of density over the given surface, but we need to determine the values of x and y, so we need to find the projection R of the lamina onto the xy-plane. Taking $z = 0$ gives the line $x + y = 1$, so R is formed by this line together with the coordinate axes $x = 0$ and $y = 0$. In particular, we have

$$\text{Mass} = \iint_\sigma \delta\, dS = \int_0^1 \int_0^{1-y} \delta\sqrt{3}\, dx\, dy = \delta\sqrt{3} \int_0^1 (1 - y)\, dy = \frac{\delta\sqrt{3}}{2}.$$