MA2E01 Tutorial solutions #7

1. Compute both div $\boldsymbol{F}$ and curl $\boldsymbol{F}$ in the case that $\boldsymbol{F} = \langle e^{xy}, \sin y, y \ln z\rangle$.

The divergence of $\boldsymbol{F}$ is given by

$$\operatorname{div} \boldsymbol{F} = (e^{xy})_x + (\sin y)_y + (y \ln z)_z = ye^{xy} + \cos y + y/z,$$

while the curl of $\boldsymbol{F}$ is given by

$$\begin{aligned} \operatorname{curl} \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \\ &= (\ln z)\boldsymbol{i} - xe^{xy}\boldsymbol{k}. \end{aligned}$$

2. Let C denote the line from $(1,1)$ to $(2,3)$. Compute the line integrals

$$\int_C (x+y)\, ds, \qquad \int_C x\, dy.$$

The parametric equation of the line from $(1,1)$ to $(2,3)$ is

$$\boldsymbol{r}(t) = (1-t)\cdot\langle 1,1\rangle + t\cdot\langle 2,3\rangle = \langle 1+t, 1+2t\rangle, \qquad 0 \le t \le 1.$$

This gives $\boldsymbol{r}'(t) = \langle 1,2\rangle$ and $x + y = 2 + 3t$, so the first integral is equal to

$$\int_C (x+y)\, ds = \int_0^1 (2+3t)\cdot\sqrt{5}\, dt = \left(2 + \frac{3}{2}\right)\sqrt{5} = \frac{7\sqrt{5}}{2}.$$

For the second integral, we have $dy = y'(t)\, dt = 2\, dt$ and thus

$$\int_C x\, dy = \int_0^1 (1+t)\cdot 2\, dt = 2\left(1 + \frac{1}{2}\right) = 2 + 1 = 3.$$

3. Find the work done by the force field $\boldsymbol{F} = \langle -x, y\rangle$ while moving an object from $(1,0)$ to $(0,1)$ along the part of the unit circle that lies in the first quadrant.

The given curve has equation $\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle$ with $0 \le t \le \pi/2$, so the work is

$$\begin{aligned} \int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{\pi/2} \langle -\cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{\pi/2} 2\sin t \cos t\, dt = \Big[\sin^2 t\Big]_0^{\pi/2} = 1. \end{aligned}$$

4. Show that $\boldsymbol{F} = \langle 4x^3 - 4xy, 3y^2 - 2x^2 \rangle$ is conservative and find a potential function ϕ. Use this potential function to compute the line integral

$$\int_{(1,1)}^{(2,2)} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

First, we note that $\boldsymbol{F}$ is conservative because

$$(4x^3 - 4xy)_y = -4x = (3y^2 - 2x^2)_x.$$

To find a potential function ϕ, we need to ensure that

$$\phi_x = F_1 = 4x^3 - 4xy, \qquad \phi_y = F_2 = 3y^2 - 2x^2$$

and we may integrate these equations to get

$$\phi = \int (4x^3 - 4xy)\, dx = x^4 - 2x^2 y + C_1(y),$$
$$\phi = \int (3y^2 - 2x^2)\, dy = y^3 - 2x^2 y + C_2(x).$$

This implies that $\phi(x, y) = x^4 + y^3 - 2x^2 y$ is a potential function and that

$$\int_{(1,1)}^{(2,2)} \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(2, 2) - \phi(1, 1) = 8 - 0 = 8.$$