MA2E01 Tutorial solutions #6

1. Use cylindrical coordinates to compute the volume of the solid which is bounded by the paraboloid $z = x^2 + y^2$ from below and by the plane $z = 4$ from above.

The projection onto the xy-plane is the interior of the circle $x^2 + y^2 = 4$, so

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3)\, dr\, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_{r=0}^2 d\theta = \int_0^{2\pi} (8 - 4)\, d\theta = 8\pi. \end{aligned}$$

2. Use spherical coordinates to compute the volume of the solid which is bounded by the cone $\phi = \pi/3$ from below and by the sphere $\rho = 3$ from above.

In terms of spherical coordinates, the volume of the given solid is

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} 9 \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \Big[-9\cos\phi \Big]_{\phi=0}^{\pi/3} d\theta = \int_0^{2\pi} (-9/2 + 9)\, d\theta = 9\pi. \end{aligned}$$

3. Use cylindrical coordinates to evaluate the integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} x(x^2 + y^2)\, dz\, dy\, dx.$$

In cylindrical coordinates, we have $x(x^2 + y^2) = r^3 \cos\theta$ and this implies

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} r^4 \cos\theta\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 (r^4 - r^6) \cos\theta\, dr\, d\theta \\ &= \int_0^{\pi/2} \left(\frac{1}{5} - \frac{1}{7} \right) \cos\theta\, d\theta = \frac{2\sin(\pi/2)}{35} = \frac{2}{35}. \end{aligned}$$

4. Let R be the region in the xy-plane which is bounded by the lines

$$x + y = 1, \qquad x + y = 2, \qquad y - x = 0, \qquad y - x = 2.$$

Use an appropriate change of variables to compute the integral $\iint_R (y^2 - x^2)\, dA$.

We use the change of variables $u = x + y$ and $v = y - x$, which means that

$$x = \frac{u - v}{2}, \qquad y = \frac{u + v}{2}.$$

The Jacobian of this transformation is then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

so the given integral is equal to

$$\begin{aligned} \iint_R (y^2 - x^2)\, dA &= \int_0^2 \int_1^2 \frac{uv}{2}\, du\, dv \\ &= \int_0^2 \left[\frac{u^2 v}{4}\right]_{u=1}^2 dv = \int_0^2 \frac{3v}{4}\, dv = \left[\frac{3v^2}{8}\right]_0^2 = \frac{3}{2}. \end{aligned}$$