MA2E01 Tutorial solutions #5

1. Use polar coordinates to evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(x^2+y^2)\,dy\,dx.$$

Switching to polar coordinates, we can write the given integral as

$$\int_0^{\pi/2} \int_0^1 r\cos(r^2)\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{\sin(r^2)}{2}\right]_{r=0}^1 d\theta = \int_0^{\pi/2} \frac{\sin 1}{2}\,d\theta = \frac{\pi\sin 1}{4}.$$

2. Use polar coordinates to evaluate the integral

$$\int_0^2 \int_y^{\sqrt{8-y^2}} \frac{dx\,dy}{\sqrt{1+x^2+y^2}}.$$

Switching to polar coordinates, we can write the given integral as

$$\int_0^{\pi/4} \int_0^{\sqrt{8}} \frac{r\,dr\,d\theta}{\sqrt{1+r^2}} = \int_0^{\pi/4} \left[\sqrt{1+r^2}\right]_{r=0}^{\sqrt{8}} d\theta = \int_0^{\pi/4} 2\,d\theta = \frac{\pi}{2}.$$

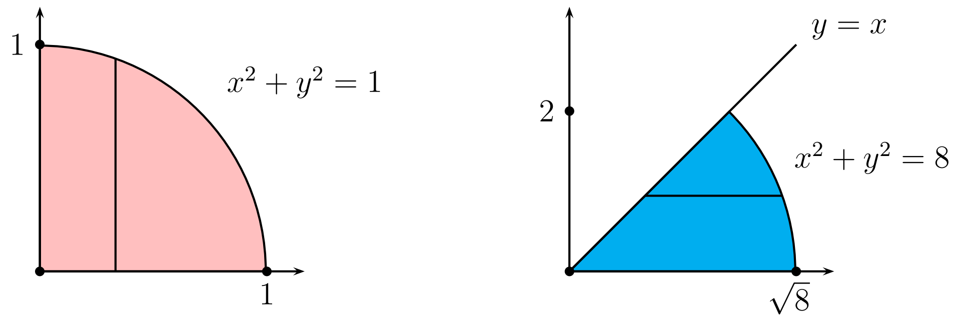

Figure: The regions of integration for Problems 1 and 2.

3. The sphere of radius a around the origin is given by the parametric equation

$$\boldsymbol{r}(\theta,\phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle,$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. Use this fact to compute its area.

To compute the area, we first need to find the normal vector $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta$, namely

$$\begin{aligned}
\boldsymbol{r}_\phi &= \langle a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi\rangle, \\
\boldsymbol{r}_\theta &= \langle -a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0\rangle, \\
\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta &= \left\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\right\rangle = a\sin\phi\cdot\boldsymbol{r}.
\end{aligned}$$

Since $\boldsymbol{r}$ lies on the sphere of radius a, we have $||\boldsymbol{r}|| = a$ and this implies

$$\begin{aligned}
\text{Surface area} &= \int_0^{2\pi}\int_0^{\pi} ||\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta||\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} \Big[-a^2\cos\phi\Big]_{\phi=0}^{\pi} d\theta = \int_0^{2\pi} 2a^2\, d\theta = 4\pi a^2.
\end{aligned}$$

4. A lamina with density $\delta(x,y) = x + y$ is bounded by the x-axis, the line $x = 1$ and the curve $y = \sqrt{x}$. Find its mass and also its center of gravity.

The mass of the lamina is

$$\begin{aligned}
M &= \int_0^1\int_0^{\sqrt{x}} (x+y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x^{3/2} + \frac{x}{2}\right) dx = \left[\frac{2x^{5/2}}{5} + \frac{x^2}{4}\right]_0^1 = \frac{13}{20}.
\end{aligned}$$

Its center of gravity is the point (x_0, y_0), where x_0 is given by

$$\begin{aligned}
x_0 &= \frac{20}{13}\int_0^1\int_0^{\sqrt{x}} x(x+y)\, dy\, dx = \frac{20}{13}\int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_{y=0}^{\sqrt{x}} dx \\
&= \frac{20}{13}\int_0^1 \left(x^{5/2} + \frac{x^2}{2}\right) dx = \frac{20}{13}\left[\frac{2x^{7/2}}{7} + \frac{x^3}{6}\right]_0^1 = \frac{190}{273}
\end{aligned}$$

and y_0 is similarly given by

$$\begin{aligned}
y_0 &= \frac{20}{13}\int_0^1\int_0^{\sqrt{x}} y(x+y)\, dy\, dx = \frac{20}{13}\int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{y=0}^{\sqrt{x}} dx \\
&= \frac{20}{13}\int_0^1 \left(\frac{x^2}{2} + \frac{x^{3/2}}{3}\right) dx = \frac{20}{13}\left[\frac{x^3}{6} + \frac{2x^{5/2}}{15}\right]_0^1 = \frac{6}{13}.
\end{aligned}$$