MA2E01 Tutorial solutions #4

1. Find the critical points of the function

$$f(x,y) = xy - x^3 - y^2 + x$$

and then classify them as relative minima, relative maxima or saddle points.

To find the critical points, one has to solve the equations

$$0 = f_x = y - 3x^2 + 1, \qquad 0 = f_y = x - 2y.$$

Eliminating y from the first equation gives $y = 3x^2 - 1$, hence also

$$x = 2y = 6x^2 - 2 \quad \Longrightarrow \quad 6x^2 - x - 2 = 0 \quad \Longrightarrow \quad x = 2/3, -1/2.$$

The only critical points are thus $A(2/3, 1/3)$ and $B(-1/2, -1/4)$, while

$$D = f_{xx}f_{yy} - f_{xy}^2 = (-6x)(-2) - 1^2 = 12x - 1.$$

At the point A, we have $D > 0$ and $f_{xx} < 0$, so this corresponds to a local maximum. At the point B, we have $D < 0$, so this corresponds to a saddle point.

2. Compute each of the following integrals:

$$\int_1^3 \int_0^\pi y \sin x \, dx \, dy, \qquad \int_0^3 \int_0^2 y e^{xy} \, dx \, dy.$$

When it comes to the first integral, we have

$$\int_0^\pi y \sin x \, dx = \Big[-y \cos x \Big]_{x=0}^{\pi} = -y \cos \pi + y \cos 0 = 2y$$

and this implies that

$$\int_1^3 \int_0^\pi y \sin x \, dx \, dy = \int_1^3 2y \, dy = \Big[y^2 \Big]_1^3 = 3^2 - 1^2 = 8.$$

When it comes to the second integral, we similarly have

$$\int_0^2 y e^{xy} \, dx = \Big[e^{xy} \Big]_{x=0}^{2} = e^{2y} - 1$$

and this implies that

$$\int_0^3 \int_0^2 y e^{xy} \, dx \, dy = \int_0^3 (e^{2y} - 1) \, dy = \left[\frac{e^{2y}}{2} - y \right]_0^3 = \frac{e^6 - 7}{2}.$$

3. Let R be the triangular region whose vertices are $(0,0)$, $(1,0)$ and $(1,3)$. Find the volume of the solid that lies below the plane $z = 5 - x - y$ and above the region R.

In view of the figure below, the desired volume is given by

$$\begin{aligned} V &= \int_0^1 \int_0^{3x} (5 - x - y)\, dy\, dx = \int_0^1 \left[(5-x)y - \frac{y^2}{2} \right]_{y=0}^{3x} dx \\ &= \int_0^1 \left(15x - 3x^2 - \frac{9x^2}{2} \right) dx = \left[\frac{15x^2}{2} - x^3 - \frac{3x^3}{2} \right]_0^1 = 5. \end{aligned}$$

4. Let $a > 0$ be a constant. Switch the order of integration to compute the integral

$$\int_0^{2a} \int_{y/2}^{a} e^{x^2}\, dx\, dy.$$

In this case, switching the order of integration gives

$$\int_0^{2a} \int_{y/2}^{a} e^{x^2}\, dx\, dy = \int_0^a \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^a 2xe^{x^2}\, dx = e^{a^2} - 1.$$

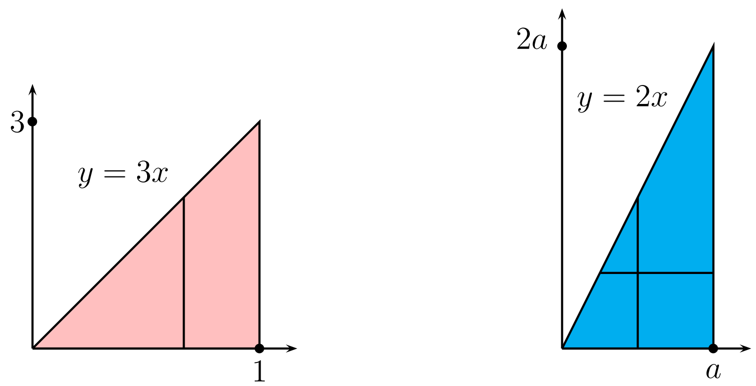

Figure: The regions of integration for Problems 3 and 4.