MA2E01 Tutorial solutions #3

1. Use the chain rule to compute the partial derivatives z_s, z_t in the case that

$$z = \ln(x^2 + y^3), \qquad x = s^2 t, \qquad y = \sin(st).$$

According to the chain rule, one has

$$z_s = z_x x_s + z_y y_s = \frac{2x}{x^2+y^3} \cdot 2st + \frac{3y^2}{x^2+y^3} \cdot t\cos(st),$$
$$z_t = z_x x_t + z_y y_t = \frac{2x}{x^2+y^3} \cdot s^2 + \frac{3y^2}{x^2+y^3} \cdot s\cos(st).$$

2. Consider the function $f(x,y,z) = x^2 e^{y/z}$ at the point $(3,0,1)$.

(a) What is the rate at which f is changing in the direction of $\boldsymbol{u} = \langle 1,2,1\rangle$?

The gradient at the given point is

$$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle 2xe^{y/z}, \frac{x^2 e^{y/z}}{z}, -\frac{x^2 y e^{y/z}}{z^2} \right\rangle = \langle 6, 9, 0\rangle\,.$$

Noting that the vector $\boldsymbol{u}$ has length $||\boldsymbol{u}|| = \sqrt{6}$, we conclude that

$$D_{\boldsymbol{u}} f = \frac{1}{\sqrt{6}}\, \boldsymbol{u} \cdot \nabla f = \frac{6+18}{\sqrt{6}} = \frac{24\sqrt{6}}{6} = 4\sqrt{6}.$$

(b) Find a unit vector in the direction in which f increases most rapidly.

The direction of most rapid increase is that of the gradient $\nabla f = \langle 6,9,0\rangle$. To find a unit vector $\boldsymbol{v}$ in the same direction, we simply divide by the length of ∇f to get

$$||\nabla f|| = \sqrt{6^2+9^2} = 3\sqrt{13} \quad\Longrightarrow\quad \boldsymbol{v} = \left\langle 2/\sqrt{13}, 3/\sqrt{13}, 0 \right\rangle.$$

3. Consider the function $f(x,y) = \sqrt{\sin(x^2 y) + x^3 + 2y}$ at the point $(1,0)$.

(a) Find the direction in which f is decreasing most rapidly.

The direction of most rapid decrease is that of $-\nabla f = -\langle f_x, f_y\rangle$. In this case,

$$f_x = \frac{\cos(x^2y)(x^2y)_x + 3x^2}{2\sqrt{\sin(x^2y) + x^3 + 2y}} = \frac{2xy\cos(x^2y) + 3x^2}{2\sqrt{\sin(x^2y) + x^3 + 2y}}$$

and we can similarly compute

$$f_y = \frac{\cos(x^2y)(x^2y)_y + 2}{2\sqrt{\sin(x^2y) + x^3 + 2y}} = \frac{x^2\cos(x^2y) + 2}{2\sqrt{\sin(x^2y) + x^3 + 2y}}.$$

Once we now substitute $x = 1$ and $y = 0$, we get the vector

$$-\nabla f = -\langle f_x, f_y \rangle = -\langle 3/2, 3/2 \rangle .$$

(b) Find the equation of the tangent plane at the given point.

At the given point, we have $f_x = f_y = 3/2$ by above, so the tangent plane is

$$z - z_0 = \frac{3}{2}(x - x_0) + \frac{3}{2}(y - y_0).$$

Since $x_0 = 1$ and $y_0 = 0$, we have $z_0 = f(1, 0) = 1$ and this gives

$$z = \frac{3}{2}(x - 1) + \frac{3y}{2} + 1 = \frac{3x + 3y - 1}{2}.$$

4. Suppose that $z = f(x - y, y - x)$ for some function f. Show that $z_x + z_y = 0$.

In this case, we have $z = f(u, v)$, where $u = x - y$ and $v = y - x$, so

$$z_x = z_u u_x + z_v v_x = z_u - z_v,$$
$$z_y = z_u u_y + z_v v_y = -z_u + z_v.$$

Adding these two equations, we conclude that $z_x + z_y = 0$.