MA2E01 Tutorial solutions #10

1. Solve $y''(t) - y(t) = \sin t$ subject to the conditions $y(0) = 0$ and $y'(0) = 2$.

Applying the Laplace transform to both sides gives

$$s^2 \mathscr{L}(y) - sy(0) - y'(0) - \mathscr{L}(y) = \frac{1}{s^2+1}$$

and we can write this equation as

$$(s^2 - 1) \cdot \mathscr{L}(y) = 2 + \frac{1}{s^2+1} = \frac{2s^2+3}{s^2+1}.$$

Next, we divide by $s^2 - 1$ and use partial fractions to find that

$$\mathscr{L}(y) = \frac{2s^2+3}{(s^2-1)(s^2+1)} = \frac{5/4}{s-1} - \frac{5/4}{s+1} - \frac{1/2}{s^2+1}.$$

Using this fact and our table of Laplace transforms, we conclude that

$$y(t) = \frac{5e^t}{4} - \frac{5e^{-t}}{4} - \frac{\sin t}{2}.$$

2. Solve $y''(t) - 2y'(t) - 3y(t) = t$ subject to the conditions $y(0) = 1$ and $y'(0) = 2$.

Once again, we apply the Laplace transform to get

$$s^2 \mathscr{L}(y) - sy(0) - y'(0) - 2\Big(s\mathscr{L}(y) - y(0)\Big) - 3\mathscr{L}(y) = \frac{1}{s^2}$$

and then simplify this equation to arrive at

$$(s^2 - 2s - 3) \cdot \mathscr{L}(y) = s + \frac{1}{s^2} = \frac{s^3+1}{s^2}.$$

Noting that $s^2 - 2s - 3 = (s-3)(s+1)$, we employ partial fractions to find that

$$\mathscr{L}(y) = \frac{s^3+1}{s^2(s-3)(s+1)} = \frac{2s/9}{s^2} - \frac{1/3}{s^2} + \frac{7/9}{s-3} = \frac{2/9}{s} - \frac{1/3}{s^2} + \frac{7/9}{s-3}.$$

Using this fact and our table of Laplace transforms, we conclude that

$$y(t) = \frac{2}{9} - \frac{t}{3} + \frac{7e^{3t}}{9}.$$

3. Solve $y''(t) - 4y(t) = 5e^{2t}$ subject to the conditions $y(0) = 0$ and $y'(0) = 1$.

Once again, we apply the Laplace transform to get

$$s^2\mathscr{L}(y) - sy(0) - y'(0) - 4\mathscr{L}(y) = \frac{5}{s-2}$$

and then simplify this equation to arrive at

$$(s^2 - 4) \cdot \mathscr{L}(y) = 1 + \frac{5}{s-2} = \frac{s+3}{s-2}.$$

Noting that $s^2 - 4 = (s-2)(s+2)$, we employ partial fractions to find that

$$\mathscr{L}(y) = \frac{s+3}{(s-2)^2(s+2)} = \frac{1/16}{s+2} - \frac{(s-22)/16}{(s-2)^2}.$$

For the rightmost term, the denominator $s - 2$ must be shifted to s and this gives

$$\begin{aligned} y(t) &= \frac{e^{-2t}}{16} - \frac{1}{16}\mathscr{L}^{-1}\left(\frac{s-22}{(s-2)^2}\right) = \frac{e^{-2t}}{16} - \frac{e^{2t}}{16}\mathscr{L}^{-1}\left(\frac{s+2-22}{s^2}\right) \\ &= \frac{e^{-2t}}{16} - \frac{e^{2t}}{16}\mathscr{L}^{-1}\left(\frac{1}{s} - \frac{20}{s^2}\right) = \frac{e^{-2t}}{16} - \frac{e^{2t}(1-20t)}{16}. \end{aligned}$$

4. Solve $y''(t) + 4y(t) = u(t - \pi)$ subject to the conditions $y(0) = 2$ and $y'(0) = 1$.

Once again, we apply the Laplace transform to get

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + 4\mathscr{L}(y) = \frac{e^{-\pi s}}{s}$$

and then simplify this equation to arrive at

$$(s^2 + 4) \cdot \mathscr{L}(y) = 2s + 1 + \frac{e^{-\pi s}}{s}.$$

Using this fact and partial fractions, we now find that

$$\mathscr{L}(y) = \frac{2s+1}{s^2+4} + \frac{e^{-\pi s}}{s(s^2+4)} = \frac{2s+1}{s^2+4} + \frac{e^{-\pi s}}{4s} - \frac{se^{-\pi s}}{4(s^2+4)}.$$

If we ignore the exponential factor for the moment, then our table gives

$$\mathscr{L}^{-1}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) = \frac{1}{4} - \frac{\cos(2t)}{4}.$$

If we include the exponential factor, then t becomes $t - \pi$ and we finally get

$$\begin{aligned} y(t) &= 2\cos(2t) + \frac{\sin(2t)}{2} + u(t-\pi) \cdot \left(\frac{1}{4} - \frac{\cos(2t - 2\pi)}{4}\right) \\ &= 2\cos(2t) + \frac{\sin(2t)}{2} + u(t-\pi) \cdot \frac{1 - \cos(2t)}{4}. \end{aligned}$$

5. Solve $y''(t) + y(t) = u(t-\pi) + \delta(t-2\pi)$ subject to the conditions $y(0) = y'(0) = 1$.

Once again, we apply the Laplace transform to get

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + \mathscr{L}(y) = \frac{e^{-\pi s}}{s} + e^{-2\pi s}$$

and then simplify this equation to arrive at

$$(s^2+1)\cdot\mathscr{L}(y) = s + 1 + \frac{e^{-\pi s}}{s} + e^{-2\pi s}.$$

Using this fact and partial fractions, we now find that

$$\begin{aligned}\mathscr{L}(y) &= \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)} + \frac{e^{-2\pi s}}{s^2+1} \\ &= \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s} - \frac{se^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}.\end{aligned}$$

Once we now apply the inverse Laplace transform, we may finally conclude that

$$\begin{aligned}y(t) &= \cos t + \sin t + u(t-\pi) - u(t-\pi)\cos(t-\pi) + u(t-2\pi)\sin(t-2\pi) \\ &= \cos t + \sin t + u(t-\pi) + u(t-\pi)\cos t + u(t-2\pi)\sin t.\end{aligned}$$