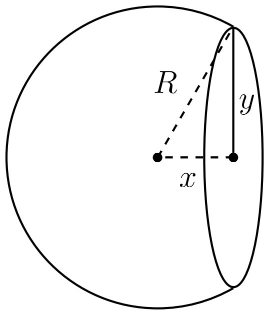
MA2E01 Tutorial solutions #1

1. Compute the volume of a sphere of radius R using the method of slicing.

Place the sphere along the x-axis so that its center lies at the origin. Then the cross section of the sphere at each point $-R \leq x \leq R$ is a circle of radius $y = \sqrt{R^2 - x^2}$.



Since the cross section has area $\pi y^2 = \pi(R^2 - x^2)$, the volume of the sphere is

$$\text{Volume} = \int_{-R}^{R} \pi(R^2 - x^2)\, dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R} = \frac{4\pi R^3}{3}.$$

2. Does the straight line through $A(1, 2, 3)$ and $B(3, 3, 7)$ pass through $C(7, 6, 9)$?

If it does, then the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ must describe the same direction. Since

$$\overrightarrow{AB} = \langle 2, 1, 4 \rangle, \qquad \overrightarrow{BC} = \langle 4, 3, 2 \rangle$$

are not scalar multiples of one another, however, this is not the case.

3. Are the points $A(1, 0, 2)$, $B(5, 3, 4)$ and $C(3, -4, 4)$ the vertices of a right triangle?

To check if the angle at the point A is a right angle, we compute the dot product

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \langle 4, 3, 2 \rangle \cdot \langle 2, -4, 2 \rangle = 8 - 12 + 4 = 0.$$

As this product is zero, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular to one another.

4. The position of a moving object is given by the vector-valued function

$$\boldsymbol{r}(t) = \left\langle t^3 + t,\ 2 - \ln t,\ \sin(\pi t) \right\rangle.$$

(a) Find the velocity of this object at any given time $t > 0$.

The velocity is given by the derivative $\boldsymbol{r}'(t) = \langle 3t^2 + 1, -1/t, \pi \cos(\pi t) \rangle$.

(b) Find the equation of the tangent line to the curve at time $t = 1$.

Since $\boldsymbol{r}(1) = \langle 2, 2, 0\rangle$ and $\boldsymbol{r}'(1) = \langle 4, -1, -\pi\rangle$, the equation of the tangent line is

$$x = 2 + 4t, \qquad y = 2 - t, \qquad z = -\pi t.$$

(c) Does the object move faster when $t = 1$ or when $t = 2$?

We note that $\boldsymbol{r}'(1) = \langle 4, -1, -\pi\rangle$ and $\boldsymbol{r}'(2) = \langle 13, -1/2, \pi\rangle$, while

$$||\boldsymbol{r}'(1)|| = \sqrt{4^2 + 1^2 + \pi^2}, \qquad ||\boldsymbol{r}'(2)|| = \sqrt{13^2 + (1/2)^2 + \pi^2}.$$

Since the latter is larger than the former, the object was moving faster when $t = 2$.