Lecture 1, September 24

- **Area between two graphs.** If the graph of f lies above the graph of g, then the area that lies between the two graphs from $x = a$ to $x = b$ is given by

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx.$$

- **Volumes by slicing.** A solid is placed along the x-axis between $x = a$ and $x = b$. If its cross section at each point x has area $A(x)$, then the volume of the solid is

$$\text{Volume} = \int_a^b A(x)\, dx.$$

. . . .

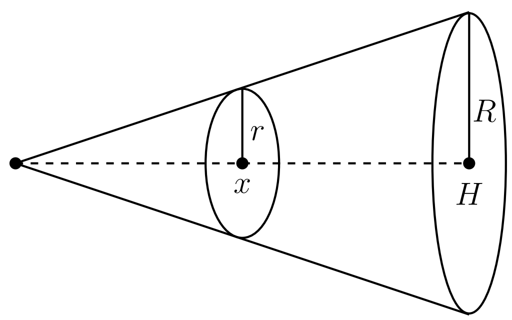
Example 1. Let $f(x) = 4x$ and $g(x) = x^2$. To compute the area of the region that lies between the two graphs, we note that the graphs intersect when

$$4x = x^2 \quad \Longrightarrow \quad x(x-4) = 0 \quad \Longrightarrow \quad x = 0, 4.$$

Since a quick sketch shows that the graph of f lies above the graph of g between the two points of intersection, the area of the desired region is

$$\text{Area} = \int_0^4 (4x - x^2)\, dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}.$$

Example 2. Consider a cone of radius R and height H. To place such a cone along the x-axis, we put its vertex at the origin and the center of its base at $(H, 0)$.



The cross section of the cone at each point x is then a circle of radius r, where

$$\frac{r}{x} = \frac{R}{H} \quad \Longrightarrow \quad r = \frac{Rx}{H}$$

by similar triangles. Since the cross section has area $\pi r^2 = \pi R^2 x^2 / H^2$, we get

$$\text{Volume} = \int_0^H \frac{\pi R^2 x^2}{H^2}\, dx = \frac{\pi R^2 H^3}{3H^2} = \frac{\pi R^2 H}{3}.$$

Lecture 2, September 26

- **Vectors.** The vector that points from $A(a_1, a_2, a_3)$ to $B(b_1, b_2, b_3)$ is given by

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle .$$

Vector addition and scalar multiplication are defined component-wise:

$$\overrightarrow{v} + \overrightarrow{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle , \qquad \lambda \overrightarrow{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$$

whenever $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$, $\overrightarrow{w} = \langle w_1, w_2, w_3 \rangle$ and λ is a scalar. Also, the expression

$$||\overrightarrow{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

gives the length (or norm) of $\overrightarrow{v}$, while similar formulas hold for vectors in $\mathbb{R}^2$.

- **Dot product.** The dot product of two vectors may be computed in two ways:

$$\overrightarrow{v} \cdot \overrightarrow{w} = v_1 w_1 + v_2 w_2 + v_3 w_3, \qquad \overrightarrow{v} \cdot \overrightarrow{w} = ||\overrightarrow{v}|| \cdot ||\overrightarrow{w}|| \cdot \cos\theta.$$

Here, the two vectors have the same starting point and θ is the angle between them.

- **Parallel/orthogonal.** Two vectors are parallel if and only if they are scalar multiples of one another. Two vectors are orthogonal if and only if their dot product is zero.

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Example 1. Consider the triangle with vertices $A(1, 0, 2)$, $B(5, 3, 4)$ and $C(3, -4, 4)$. To see that the angle at the point A is a right angle, we note that

$$\overrightarrow{AB} = \langle 4, 3, 2 \rangle , \qquad \overrightarrow{AC} = \langle 2, -4, 2 \rangle \quad \Longrightarrow \quad \overrightarrow{AB} \cdot \overrightarrow{AC} = 8 - 12 + 4 = 0.$$

To determine the angle at the point B, we use the formula

$$\cos B = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{||\overrightarrow{BA}|| \cdot ||\overrightarrow{BC}||} = \frac{\langle -4, -3, -2 \rangle \cdot \langle -2, -7, 0 \rangle}{\sqrt{4^2 + 3^2 + 2^2}\sqrt{2^2 + 7^2}} = \frac{29}{\sqrt{29}\sqrt{53}} = \sqrt{\frac{29}{53}}.$$

Example 2. Consider the points $A(1, 2, 3)$, $B(4, 2, 1)$ and $C(1, 1, 1)$. The vectors

$$\overrightarrow{AB} = \langle 3, 0, -2 \rangle , \qquad \overrightarrow{BC} = \langle -3, -1, 0 \rangle$$

are not parallel, as they are not scalar multiples of one another. This means that they describe different directions, so the three given points are not collinear.

Lecture 3, September 28

- **Lines.** The line that passes through $A(a_1, a_2, a_3)$ with direction $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ can be described using the parametric equations

$$x = a_1 + tv_1, \qquad y = a_2 + tv_2, \qquad z = a_3 + tv_3.$$

- **Vector-valued functions.** A vector-valued function is one that has the form

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle .$$

  Its limits, derivatives and integrals may all be computed component-wise. Its graph is a curve in $\mathbb{R}^3$ and its derivative $\boldsymbol{r}'(t)$ is tangent to the curve at each point. If $\boldsymbol{r}(t)$ is the position of a moving object, then $\boldsymbol{r}'(t)$ is its velocity and $||\boldsymbol{r}'(t)||$ is its speed.

- **Tangent line.** The tangent line to the curve $\boldsymbol{r}(t)$ at time $t = t_0$ passes through the point $\boldsymbol{r}(t_0)$ with direction $\boldsymbol{r}'(t_0)$. One may determine its equation as above.

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Example 1. Consider the points $A(1, 2, 4)$ and $B(5, 3, 2)$. To determine the line that passes through these points, we note that the direction of this line is

$$\overrightarrow{AB} = \langle 5 - 1, 3 - 2, 2 - 4 \rangle = \langle 4, 1, -2 \rangle .$$

Since the line passes through $A(1, 2, 4)$ with that direction, its equation is then

$$x = 1 + 4t, \qquad y = 2 + t, \qquad z = 4 - 2t.$$

Example 2. Suppose that the position of a moving object is $\boldsymbol{r}(t) = \langle t^3, 2t^2, 5t \rangle$. Then its velocity vector is $\boldsymbol{r}'(t) = \langle 3t^2, 4t, 5 \rangle$. To find the tangent line when $t = 1$, we note that it passes through $(1, 2, 5)$ with direction $\boldsymbol{r}'(1) = \langle 3, 4, 5 \rangle$, so its equation is

$$x = 1 + 3t, \qquad y = 2 + 4t, \qquad z = 5 + 5t.$$

To compute the object's speed at time $t = 1$, we note that

$$\boldsymbol{r}'(1) = \langle 3, 4, 5 \rangle \quad \Longrightarrow \quad ||\boldsymbol{r}'(1)|| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}.$$

Lecture 4, October 1

- **Cross product.** The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\boldsymbol{v} \times \boldsymbol{w} = \langle v_2w_3 - v_3w_2,\, v_3w_1 - v_1w_3,\, v_1w_2 - v_2w_1 \rangle .$$

This vector is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$, while its length is

$$||\boldsymbol{v} \times \boldsymbol{w}|| = ||\boldsymbol{v}|| \cdot ||\boldsymbol{w}|| \cdot \sin\theta.$$

- **Normal vector.** We say that the vector $\boldsymbol{n}$ is normal to a plane, if $\boldsymbol{n}$ is orthogonal to every vector that lies on the plane. If a plane passes through $A(a_1, a_2, a_3)$ and its normal vector is $\boldsymbol{n} = \langle n_1, n_2, n_3 \rangle$, then its equation is

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0.$$

....

Example 1. Consider the points $A(2, 3, 4)$, $B(1, 0, 2)$ and $C(3, 2, 1)$. To find the plane that passes through these points, we note that its normal vector is

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -1, -3, -2 \rangle \times \langle 1, -1, -3 \rangle = \langle 7, -5, 4 \rangle .$$

Since the plane passes through $A(2, 3, 4)$, its equation is then

$$7(x - 2) - 5(y - 3) + 4(z - 4) = 0 \quad \Longrightarrow \quad 7x - 5y + 4z = 15.$$

Example 2. Consider the line through $P(2, 4, 1)$ and $Q(4, 1, 5)$. To find the point at which it intersects the plane $x - 2y + 3z = 37$, we first find the equation of the line. Since $\overrightarrow{PQ} = \langle 2, -3, 4 \rangle$ is the direction of the line, its equation is

$$x = 2 + 2t, \qquad y = 4 - 3t, \qquad z = 1 + 4t.$$

The point we wish to find is the point which satisfies this equation (because it is on the line) as well as $x - 2y + 3z = 37$ (because it is on the plane). This gives

$$(2 + 2t) - 2(4 - 3t) + 3(1 + 4t) = 37 \quad \Longrightarrow \quad 20t = 40 \quad \Longrightarrow \quad t = 2,$$

so the point of intersection is the point

$$(x, y, z) = (2 + 2t, 4 - 3t, 1 + 4t) = (6, -2, 9).$$

Lecture 5, October 3

- **Functions of two variables.** The domain of a function $z = f(x, y)$ is the set of all points (x, y) at which it is defined. The graph of such a function is a surface in $\mathbb{R}^3$. To draw a rough sketch of the graph, one looks at the level curves $f(x, y) = k$ for various values of k. These are curves in the xy-plane that correspond to horizontal slices of the graph; they describe the part of the graph which lies at height $z = k$.

. . . .

Example 1. Consider the function $z = x^2 + y^2$. Then the level curve $z = 1$ is

$$z = 1 \quad \Longrightarrow \quad x^2 + y^2 = 1,$$

the circle of radius 1 around the origin. One can easily draw this in the xy-plane, as we do below in the left part of the figure. Similarly, the level curve $z = 4$ is

$$z = 4 \quad \Longrightarrow \quad x^2 + y^2 = 4,$$

the circle of radius 2 around the origin. To get a rough sketch of the graph, recall that the first circle is the horizontal slice at $z = 1$, while the second circle is the horizontal slice at $z = 4$. Imagine lifting the first one up by 1 unit and the second one up by 4 units. When lifted, the circles around the origin become circles around the z-axis and the overall shape of the graph is the one depicted below.

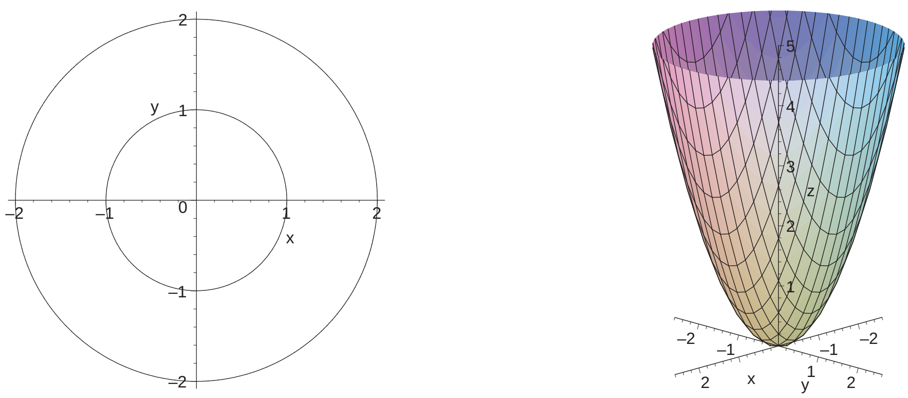

Figure 1: The level curves $x^2 + y^2 = k$ and the graph of $z = x^2 + y^2$.

Lecture 6, October 5

- **Partial derivatives.** Given a function $f(x, y)$ of two variables, we define f_x to be its derivative with respect to x when y is treated as a constant. The partial derivative f_x gives the rate at which f is changing in the x-direction. The partial derivative f_y is defined similarly by differentiating with respect to y, while treating x as a constant.

- **Mixed partials.** If the mixed partial derivatives f_{xy} and f_{yx} are continuous, then they must be equal to one another. This is the case for all standard functions.

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Example 1. Let $f(x, y) = x^2y$. Then $f_x = 2xy$ and $f_y = x^2$. The mixed partials are

$$f_{xy} = (2xy)_y = 2x, \qquad f_{yx} = (x^2)_x = 2x.$$

Example 2. Let $f(x, y) = xy^2 + 2xy + y^2$. Then we have

$$f_x = y^2 + 2y, \qquad f_y = 2xy + 2x + 2y, \qquad f_{xy} = f_{yx} = 2y + 2.$$

Example 3. Let $f(x, y) = \sin(x^2y)$. Using the chain rule, one finds that

$$\begin{aligned} f_x(x, y) &= \cos(x^2y) \cdot (x^2y)_x = \cos(x^2y) \cdot 2xy, \\ f_y(x, y) &= \cos(x^2y) \cdot (x^2y)_y = \cos(x^2y) \cdot x^2. \end{aligned}$$

Example 4. Let $f(x, y) = y \sin(xy)$. To compute f_x, we argue as before to get

$$f_x(x, y) = y \cos(xy) \cdot (xy)_x = y^2 \cos(xy).$$

To compute f_y, however, one needs to resort to the product rule; this gives

$$f_y(x, y) = \sin(xy) + y \cos(xy) \cdot (xy)_y = \sin(xy) + xy \cos(xy).$$

Example 5. Let $f(x, y, z) = (x^3 + 2y^2 + 3z)^4$. Then the first-order derivatives are

$$\begin{aligned} f_x(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 3x^2 = 12x^2(x^3 + 2y^2 + 3z)^3, \\ f_y(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 4y = 16y(x^3 + 2y^2 + 3z)^3, \\ f_z(x, y, z) &= 4(x^3 + 2y^2 + 3z)^3 \cdot 3 = 12(x^3 + 2y^2 + 3z)^3. \end{aligned}$$

In this case, equality of mixed partials means that

$$f_{xy} = f_{yx}, \qquad f_{xz} = f_{zx}, \qquad f_{yz} = f_{zy}.$$

Lecture 7, October 8

- **Chain rule.** Suppose that $f(x, y)$ depends on two variables, each of which depends on a third variable t. Then the derivative f_t is the sum of two terms, namely

$$f_t = f_x x_t + f_y y_t.$$

Similar formulas hold for functions of three or more variables; for instance,

$$f = f(x, y, z) \quad \Longrightarrow \quad f_t = f_x x_t + f_y y_t + f_z z_t.$$

- **Implicit differentiation.** Let $F(x, y, z) = 0$ be a relation between three variables. If we view z as a function of x and y, then its partial derivatives are

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

. . . .

Example 1. Let $z = x^2 y$, where $x = \sin t + 2s$ and $y = e^t + s^2 t$. Then we have

$$\begin{aligned} z_t &= z_x x_t + z_y y_t = 2xy \cdot \cos t + x^2 \cdot (e^t + s^2), \\ z_s &= z_x x_s + z_y y_s = 2xy \cdot 2 + x^2 \cdot 2st. \end{aligned}$$

Example 2. Let $z = e^{xy}$, where $x = u/v$ and $y = u^2 + 3v$. In this case,

$$\begin{aligned} z_u &= z_x x_u + z_y y_u = ye^{xy} \cdot (1/v) + xe^{xy} \cdot 2u, \\ z_v &= z_x x_v + z_y y_v = ye^{xy} \cdot (-u/v^2) + xe^{xy} \cdot 3. \end{aligned}$$

Example 3. Let $w = x^2 yz^3$, where $x = 1 + t^2$, $y = 2 - t$ and $z = 2 - t^3$. Then

$$\begin{aligned} w_t &= w_x x_t + w_y y_t + w_z z_t \\ &= 2xyz^3 \cdot 2t + x^2 z^3 \cdot (-1) + 3x^2 yz^2 \cdot (-3t^2). \end{aligned}$$

At time $t = 1$, for instance, we have $x = 2$ and $y = z = 1$, so

$$w_t = 4xyz^3 - x^2 z^3 - 9x^2 yz^2 = 8 - 4 - 36 = -32.$$

Example 4. Suppose x, y, z are related by the formula $xy^2 + xz^2 + yz = 0$. Then

$$\begin{aligned} z_x &= -\frac{(xy^2 + xz^2 + yz)_x}{(xy^2 + xz^2 + yz)_z} = -\frac{y^2 + z^2}{2xz + y}, \\ z_y &= -\frac{(xy^2 + xz^2 + yz)_y}{(xy^2 + xz^2 + yz)_z} = -\frac{2xy + z}{2xz + y}. \end{aligned}$$

Lecture 8, October 10

- **Directional derivative.** We denote by $D_{\boldsymbol{u}}f(x_0, y_0)$ the rate at which the function f changes at the point (x_0, y_0) in the direction of the unit vector $\boldsymbol{u} = \langle a, b\rangle$; that is,
$$D_{\boldsymbol{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).$$
If the vector $\boldsymbol{u}$ does not have unit length, one may simply divide it by its length.

- **Gradient vector.** Given a function $f(x, y)$ of two variables, we define its gradient to be the vector $\nabla f(x, y) = \langle f_x, f_y\rangle$. Using this notation, one can write
$$D_{\boldsymbol{u}}f(x_0, y_0) = \boldsymbol{u} \cdot \nabla f(x_0, y_0).$$
The gradient vector ∇f gives the direction of most rapid increase at each point and the rate of change in that direction is $||\nabla f||$. Similarly, $-\nabla f$ gives the direction of most rapid decrease at each point and the rate of change in that direction is $-||\nabla f||$.

- **Functions of more variables.** When $f = f(x, y, z)$, for instance, one has
$$\nabla f = \langle f_x, f_y, f_z\rangle\,, \qquad D_{\boldsymbol{u}}f = \boldsymbol{u} \cdot \nabla f$$
and the gradient vector ∇f has the exact same interpretation as before.

. . . .

Example 1. Let $f(x, y) = 3x^2 - 4xy^2$. When it comes to the point $(1, 1)$, we have
$$\nabla f = \langle f_x, f_y\rangle = \left\langle 6x - 4y^2, -8xy\right\rangle = \langle 2, -8\rangle\,.$$
Thus, the directional derivative of f in the direction of $\boldsymbol{u} = \langle 3/5, 4/5\rangle$ is
$$D_{\boldsymbol{u}}f = \boldsymbol{u} \cdot \nabla f = \frac{6}{5} - \frac{32}{5} = -\frac{26}{5}.$$

Example 2. Consider the function $f(x, y, z) = xyz^2$ at the point $(1, 2, 1)$. Then
$$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle yz^2, xz^2, 2xyz\right\rangle = \langle 2, 1, 4\rangle$$
gives the direction of most rapid increase and the corresponding rate of change is
$$||\nabla f|| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}.$$
To find the rate of change in the direction of $\boldsymbol{v} = \langle 2, 1, 2\rangle$, we note that
$$||\boldsymbol{v}|| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3,$$
so $\boldsymbol{v}$ is not a unit vector. Since $\boldsymbol{w} = \frac{1}{3}\boldsymbol{v} = \langle 2/3, 1/3, 2/3\rangle$ is a unit vector, we get
$$D_{\boldsymbol{v}}f = D_{\boldsymbol{w}}f = \boldsymbol{w} \cdot \nabla f = \frac{4}{3} + \frac{1}{3} + \frac{8}{3} = \frac{13}{3}.$$

Lecture 9, October 12

- **Tangent plane.** To find the tangent plane of a surface at a given point, one needs to find its normal vector $\boldsymbol{n}$. When $z = f(x, y)$ is given in terms of x and y, we have

$$\boldsymbol{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle .$$

When z is given implicitly in terms of a relation $F(x, y, z) = 0$, we have

$$\boldsymbol{n} = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle .$$

In either case, the normal line to the surface is the line whose direction is $\boldsymbol{n}$.

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Example 1. Consider the function $z = (x^2 + 3y)^2$ at the point $(1, 2)$. In this case,

$$\boldsymbol{n} = \big\langle 2(x^2 + 3y) \cdot 2x, 2(x^2 + 3y) \cdot 3, -1\big\rangle = \langle 28, 42, -1\rangle$$

and the tangent plane passes through $(1, 2, 49)$, so its equation is

$$28(x - 1) + 42(y - 2) - (z - 49) = 0 \quad\Longrightarrow\quad 28x + 42y - z = 63.$$

The normal line passes through the same point with direction $\boldsymbol{n}$, so it is given by

$$x = 1 + 28t, \qquad y = 2 + 42t, \qquad z = 49 - t.$$

Example 2. Consider the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2, 1, 2)$. Then

$$F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$$

and the normal vector to the tangent plane is

$$\boldsymbol{n} = \langle F_x, F_y, F_z\rangle = \langle 2x, 2y, 2z\rangle = \langle 4, 2, 4\rangle .$$

Since the tangent plane passes through $(2, 1, 2)$, its equation is

$$4(x - 2) + 2(y - 1) + 4(z - 2) = 0 \quad\Longrightarrow\quad 2x + y + 2z = 9.$$

Lecture 10, October 15

- **Critical points.** Suppose (x_0, y_0) is a critical point of $f(x, y)$ in the sense that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

To determine the behaviour of f at that point, one looks at the expression

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.$$

(a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
(b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
(c) If $D < 0$, then f has a saddle point at (x_0, y_0).

. . . .

Example 1. Let $f(x, y) = x^2 - xy + y^2 - 2x - 2y$. Then the critical points satisfy

$$0 = f_x(x, y) = 2x - y - 2, \qquad 0 = f_y(x, y) = -x + 2y - 2.$$

We multiply the first equation by 2 and then add it to the second equation to get

$$0 = 3x - 6 \quad \Longrightarrow \quad x = 2 \quad \Longrightarrow \quad y = 2x - 2 = 2.$$

This shows that $(2, 2)$ is the only critical point, while

$$D = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)^2 = 3.$$

Since $D > 0$ and $f_{xx} = 2 > 0$, the critical point $(2, 2)$ is a local minimum.

Example 2. Let $f(x, y) = 3xy - x^3 - y^3$. To find the critical points, we solve

$$0 = f_x(x, y) = 3y - 3x^2, \qquad 0 = f_y(x, y) = 3x - 3y^2.$$

These equations give $y = x^2$ and also $x = y^2$, so we easily get

$$x = y^2 = x^4 \quad \Longrightarrow \quad x^4 - x = 0 \quad \Longrightarrow \quad x = 0, 1.$$

Since $y = x^2$, the only critical points are $(0, 0)$ and $(1, 1)$, while

$$D = f_{xx} f_{yy} - f_{xy}^2 = (-6x)(-6y) - 3^2 = 36xy - 9.$$

At the critical point $(0, 0)$, we have $D = -9$ and we get a saddle point. At the critical point $(1, 1)$, we have $D > 0$ and $f_{xx} = -6x < 0$, so we get a local maximum.

Lecture 11, October 17

- **Double integrals.** The double integral of $f(x,y)$ over a region R in the xy-plane is defined in terms of Riemann sums as

$$\iint\limits_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

If f is positive, then this expression gives the volume of the solid that lies below the graph of f and above the region R in the xy-plane. When $R = [a,b]\times[c,d]$, one has

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

. . . .

Example 1. Consider $f(x,y) = x^2y$ over the rectangle $R = [0,2]\times[0,1]$. Then

$$\iint\limits_R f(x,y)\,dA = \int_0^1 \int_0^2 x^2y\,dx\,dy$$

and we can focus on the inner integral first. Integrating with respect to x, we get

$$\int_0^2 x^2y\,dx = y\int_0^2 x^2\,dx = y\left[\frac{x^3}{3}\right]_{x=0}^2 = \frac{8y}{3}$$

and so the double integral is

$$\iint\limits_R f(x,y)\,dA = \int_0^1 \frac{8y}{3}\,dy = \left[\frac{4y^2}{3}\right]_{y=0}^1 = \frac{4}{3}.$$

Alternatively, one may reach the same answer by writing

$$\iint\limits_R f(x,y)\,dA = \int_0^2 \int_0^1 x^2y\,dy\,dx$$

and by integrating with respect to y first. This approach gives

$$\int_0^1 x^2y\,dy = x^2\int_0^1 y\,dy = x^2\left[\frac{y^2}{2}\right]_{y=0}^1 = \frac{x^2}{2}$$

for the inner integral, so the double integral is equal to

$$\iint\limits_R f(x,y)\,dA = \int_0^2 \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_{x=0}^2 = \frac{4}{3}.$$

Lecture 12, October 19

- **Fubini's theorem.** If f is continuous over a bounded region R, then

$$\iint_R f(x,y)\,dA = \int\int f(x,y)\,dy\,dx = \int\int f(x,y)\,dx\,dy$$

for some suitable limits of integration that describe the region R. When it comes to the middle integral, one needs to find the possible values of y for each fixed value of x and that corresponds to a description of R using vertical slices. When it comes to the rightmost integral, one uses horizontal slices, instead.

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Example 1. Switching the order of integration, one finds that

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 ye^{y^2}\,dy = \left[\frac{e^{y^2}}{2}\right]_0^1 = \frac{e-1}{2}.$$

Example 2. We switch the order of integration in order to compute the integral

$$I = \int_0^4 \int_{y/2}^2 \frac{\cos(2y/x)}{x}\,dx\,dy = \int_0^2 \int_0^{2x} \frac{\cos(2y/x)}{x}\,dy\,dx.$$

In this case, the inner integral is given by

$$\frac{1}{x}\int_0^{2x} \cos(2y/x)\,dy = \frac{1}{x}\left[\frac{x\sin(2y/x)}{2}\right]_{y=0}^{y=2x} = \frac{\sin 4}{2}$$

and so the double integral is equal to

$$I = \int_0^2 \int_0^{2x} \frac{\cos(2y/x)}{x}\,dy\,dx = \int_0^2 \frac{\sin 4}{2}\,dx = \sin 4.$$

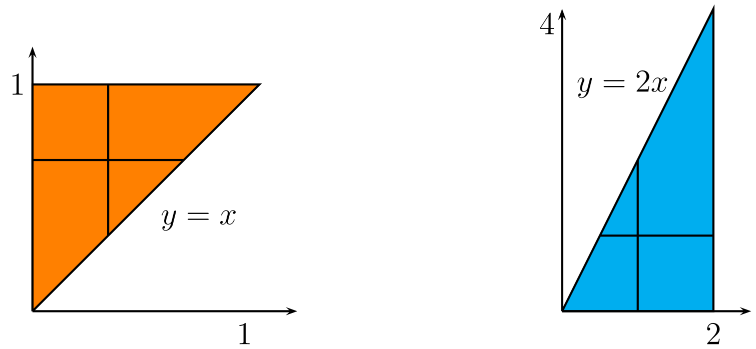


Figure: The regions of integration for Examples 1 and 2.

Lecture 13, October 22

- **Polar coordinates.** Expressing a double integral in polar coordinates, one has

$$\iint_R f(x,y)\,dA = \int\int f(r\cos\theta, r\sin\theta)\cdot r dr\,d\theta$$

for some suitable limits of integration that describe the region R.

. . . .

Example 1. If R is the region depicted on the left side of the figure, then

$$\iint_R (x^2+y^2)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

Expressing this integral in polar coordinates, one can also write it as

$$\iint_R (x^2+y^2)\,dA = \int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta.$$

Example 2. We use polar coordinates in order to compute the integral

$$I = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy.$$

In this case, the region of integration is bounded by the line $x=y$ on the left and by the circle $x=\sqrt{4-y^2}$ on the right. Note that these two intersect when

$$y=\sqrt{4-y^2} \quad\Longrightarrow\quad y^2=4-y^2 \quad\Longrightarrow\quad 2y^2=4 \quad\Longrightarrow\quad y^2=2.$$

This explains the upper limit of integration $y=\sqrt{2}$. The region of integration is thus the one depicted on the right and we can describe it using polar coordinates to get

$$I = \int_0^{\pi/4} \int_0^2 r\cdot r\,dr\,d\theta = \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^2 d\theta = \int_0^{\pi/4} \frac{8}{3}\,d\theta = \frac{2\pi}{3}.$$

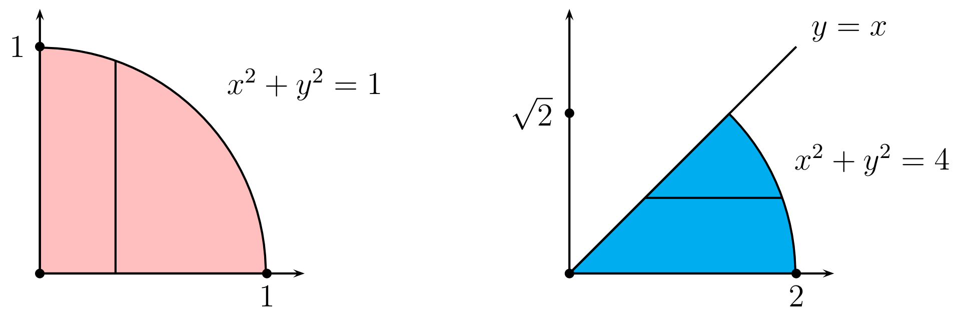


Figure: The regions of integration for Examples 1 and 2.

Lecture 14, October 24

- **Parametric surfaces.** A surface in $\mathbb{R}^3$ can be described using a vector equation

$$\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle .$$

Its normal vector is given by the cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v$, while the area of the surface which lies above the region R in the uv-plane is

$$\text{Surface area} = \iint_R ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, dA.$$

. . . .

Example 1. The parametric equation of the cone $z = \sqrt{x^2+y^2}$ is given by

$$\boldsymbol{r} = \langle x,y,z\rangle = \left\langle x, y, \sqrt{x^2+y^2}\right\rangle = \langle r\cos\theta, r\sin\theta, r\rangle .$$

Example 2. We compute the area of the part of the cylinder $x^2+y^2=1$ which lies between the planes $z=0$ and $z=3$. Its parametric equation is

$$\boldsymbol{r} = \langle \cos\theta, \sin\theta, z\rangle , \qquad 0 \le \theta \le 2\pi, \qquad 0 \le z \le 3.$$

Since the normal vector is given by

$$\boldsymbol{r}_\theta \times \boldsymbol{r}_z = \langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0,0,1\rangle = \langle \cos\theta, \sin\theta, 0\rangle ,$$

its length is equal to 1 and so the area of the cylinder is

$$\int_0^{2\pi}\int_0^3 ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z||\, dz\, d\theta = \int_0^{2\pi}\int_0^3 dz\, d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi.$$

Example 3. We compute the area of the part of the cone $z = \sqrt{x^2+y^2}$ which lies inside the cylinder $x^2+y^2=4$. Its parametric equation is

$$\boldsymbol{r} = \langle x,y,z\rangle = \langle r\cos\theta, r\sin\theta, r\rangle , \qquad 0 \le \theta \le 2\pi, \qquad 0 \le r \le 2.$$

To find its area, we first compute the normal vector

$$\boldsymbol{r}_r \times \boldsymbol{r}_\theta = \langle \cos\theta, \sin\theta, 1\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle = \langle -r\cos\theta, -r\sin\theta, r\rangle .$$

This gives $||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| = \sqrt{2r^2} = r\sqrt{2}$, so the area of the cone is

$$\int_0^{2\pi}\int_0^2 ||\boldsymbol{r}_r \times \boldsymbol{r}_\theta||\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{2}\, dr\, d\theta = \int_0^{2\pi} 2\sqrt{2}\, d\theta = 4\pi\sqrt{2}.$$

Lecture 15, October 26

- **Laminas.** If a lamina R has density function $\delta(x,y)$, then its mass is given by

$$M = \iint_R \delta(x,y)\,dA,$$

while its center of gravity is the point (x_0, y_0) whose coordinates are

$$x_0 = \frac{1}{M}\iint_R x\delta(x,y)\,dA, \qquad y_0 = \frac{1}{M}\iint_R y\delta(x,y)\,dA.$$

For laminas of constant density, the center of gravity is also known as the centroid.

- **Triple integrals.** Suppose that G is a solid which is bounded above by $z = g(x,y)$ and below by $z = h(x,y)$. If its projection onto the xy-plane is the region R, then

$$\text{Volume of } G = \iiint_G dV = \iint_R [g(x,y) - h(x,y)]\,dA.$$

....

Example 1. The lamina R inside the unit circle with $\delta(x,y) = x^2 + y^2$ has mass

$$M = \iint_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Its center of gravity (x_0, y_0) should be the origin by symmetry. In fact, we have

$$x_0 = \frac{2}{\pi}\iint_R x(x^2+y^2)\,dA = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 r^4\cos\theta\,dr\,d\theta = \frac{2}{\pi}\int_0^{2\pi}\frac{\cos\theta}{5}\,d\theta = 0$$

and a similar computation gives $y_0 = 0$ as well.

Example 2. Let G be the solid which is bounded by $z = x^2 + y^2$ from below and by the plane $z = 1$ from above. Its projection R onto the xy-plane is $x^2 + y^2 = 1$, so

$$\begin{aligned}\text{Volume of } G &= \iint_R (1 - x^2 - y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\cdot r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right)d\theta = \frac{\pi}{2}.\end{aligned}$$

Lecture 16, October 31

- **Triple integrals in cylindrical coordinates.** One has the formulas

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2+y^2 = r^2, \qquad dV = r\,dz\,dr\,d\theta.$$

- **Triple integrals in spherical coordinates.** One has the formulas

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi, \qquad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

. . . .

Example 1. Consider the solid G which is bounded by the cone $z = \sqrt{x^2+y^2}$ from below and by the sphere $x^2+y^2+z^2 = 8$ from above. To find its projection R onto the xy-plane, we find the points at which the cone meets the sphere, namely

$$x^2+y^2+z^2 = 8 \quad\Longrightarrow\quad x^2+y^2+(x^2+y^2) = 8 \quad\Longrightarrow\quad x^2+y^2 = 4.$$

This gives a circle of radius 2 in the xy-plane, so the volume of G is

$$\text{Volume} = \iiint_G dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx.$$

In terms of cylindrical coordinates, the circle $x^2+y^2 = 4$ becomes $r^2 = 4$ and so

$$\text{Volume} = \int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta.$$

In terms of spherical coordinates, finally, the equation of the cone is

$$z = \sqrt{x^2+y^2} = r \quad\Longrightarrow\quad \tan\phi = \frac{r}{z} = 1 \quad\Longrightarrow\quad \phi = \pi/4$$

and we can compute the volume of the solid as

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3\sin\phi}{3}\right]_{\rho=0}^{\sqrt{8}} d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\frac{16\sqrt{2}}{3}\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[-\frac{16\sqrt{2}}{3}\cos\phi\right]_{\phi=0}^{\pi/4} d\theta \\
&= \int_0^{2\pi}\frac{16\sqrt{2}}{3}\left(1-\frac{\sqrt{2}}{2}\right)d\theta = \frac{32\pi}{3}\cdot(\sqrt{2}-1).
\end{aligned}$$

Lecture 17, November 2

- **Formula for change of variables.** When it comes to double integrals, one has

$$\iint f(x,y)\,dx\,dy = \iint f(x(u,v),y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Here, the additional factor inside the integral is the absolute value of the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

....

Example 1. Consider the region R in the xy-plane bounded by the lines

$$x+y=1, \qquad x+y=2, \qquad x-y=0, \qquad x-y=1.$$

If we introduce the variables $u=x-y$ and $v=x+y$, then we can write

$$\iint_R \frac{x-y}{x+y}\,dx\,dy = \int_1^2\int_0^1 \frac{u}{v}\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

To compute the Jacobian in this case, we first need to solve for x and y, namely

$$\left\{\begin{array}{c} u=x-y \\ v=x+y \end{array}\right\} \implies \left\{\begin{array}{c} u+v=2x \\ v-u=2y \end{array}\right\} \implies \left\{\begin{array}{c} x=(u+v)/2 \\ y=(v-u)/2 \end{array}\right\}.$$

This allows us to differentiate x,y with respect to u,v and we now get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}.$$

Keeping this in mind, we can finally compute the given integral as

$$\begin{aligned}\iint_R \frac{x-y}{x+y}\,dx\,dy &= \int_1^2\int_0^1 \frac{u}{2v}\,du\,dv \\ &= \int_1^2 \left[\frac{u^2}{4v}\right]_{u=0}^1 dv = \int_1^2 \frac{1}{4v}\,dv = \left[\frac{\ln v}{4}\right]_1^2 = \frac{\ln 2}{4}.\end{aligned}$$

Lecture 18, November 12

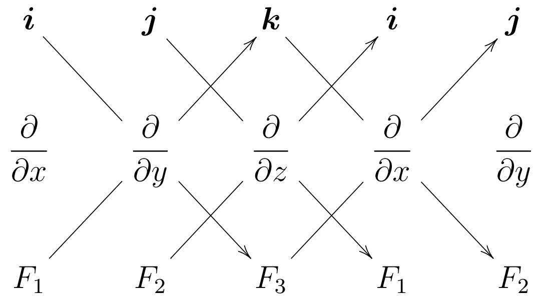
- **Divergence.** The divergence of the vector field $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$ is defined by

$$\operatorname{div} \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- **Curl.** The curl of the vector field $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$ is defined by

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \boldsymbol{k}.$$

Note that the six terms on the right hand side are the six diagonals in the diagram



with a plus sign for the diagonals going southeast and a minus sign for the others.

. . . .

Example 1. The divergence of $\boldsymbol{F}(x, y, z) = xy\boldsymbol{i} + xz\boldsymbol{j} + yz\boldsymbol{k}$ is given by

$$\operatorname{div} \boldsymbol{F} = (xy)_x + (xz)_y + (yz)_z = y + 0 + y = 2y,$$

while the curl of $\boldsymbol{F}$ is given by

$$\begin{aligned}\operatorname{curl} \boldsymbol{F} &= (yz)_y\boldsymbol{i} + (xy)_z\boldsymbol{j} + (xz)_x\boldsymbol{k} - (xy)_y\boldsymbol{k} - (xz)_z\boldsymbol{i} - (yz)_x\boldsymbol{j} \\ &= z\boldsymbol{i} + z\boldsymbol{k} - x\boldsymbol{k} - x\boldsymbol{i} \\ &= (z - x)\boldsymbol{i} + (z - x)\boldsymbol{k}.\end{aligned}$$

Example 2. Let $\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector. Then we have

$$\operatorname{div} \boldsymbol{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

and the curl is the zero vector because

$$\operatorname{curl} \boldsymbol{r} = z_y\boldsymbol{i} + x_z\boldsymbol{j} + y_x\boldsymbol{k} - x_y\boldsymbol{k} - y_z\boldsymbol{i} - z_x\boldsymbol{j} = \boldsymbol{0}.$$

Lecture 19, November 14

- **Line integrals.** The integral of $f(x, y)$ over a curve C in the xy-plane is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \cdot ||\boldsymbol{r}'(t)||\, dt,$$

where $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ is the equation of the curve and $a \leq t \leq b$. The integrals

$$\int_C f(x, y)\, dx, \qquad \int_C f(x, y)\, dy, \qquad \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

are defined similarly in terms of $dx = x'(t)\, dt$, $dy = y'(t)\, dt$ and $d\boldsymbol{r} = \boldsymbol{r}'(t)\, dt$.

. . . .

Example 1. Take $f(x, y) = xy^2$ and let C be the line from $(0, 0)$ to $(1, 2)$. Then

$$\boldsymbol{r}(t) = \langle t, 2t\rangle \quad \Longrightarrow \quad \boldsymbol{r}'(t) = \langle 1, 2\rangle \quad \Longrightarrow \quad ||\boldsymbol{r}'(t)|| = \sqrt{5}$$

and we have $0 \leq t \leq 1$. Since $x = t$ and $y = 2t$ throughout the curve, we find that

$$\int_C xy^2\, ds = \int_0^1 t(2t)^2 \cdot \sqrt{5}\, dt = 4\sqrt{5} \int_0^1 t^3\, dt = \sqrt{5}.$$

Example 2. Take $f(x, y) = x$ and let C be the part of the unit circle that lies in the first quadrant. Then we have $\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle$ with $0 \leq t \leq \pi/2$ and so

$$\boldsymbol{r}'(t) = \langle -\sin t, \cos t\rangle \quad \Longrightarrow \quad ||\boldsymbol{r}'(t)|| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

Since $x = \cos t$ by above, we conclude that

$$\int_C x\, ds = \int_0^{\pi/2} \cos t\, dt = \sin(\pi/2) - \sin 0 = 1.$$

Example 3. Take $\boldsymbol{F}(x, y) = \langle y, x\rangle$ and let C be as in the previous example. Then

$$\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle \quad \Longrightarrow \quad d\boldsymbol{r} = \langle -\sin t, \cos t\rangle\ dt$$

and also $\boldsymbol{F} = \langle y, x\rangle = \langle \sin t, \cos t\rangle$ throughout the curve, so we get

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{\pi/2} \left(-\sin^2 t + \cos^2 t\right) dt = \int_0^{\pi/2} \cos(2t)\, dt = \left[\frac{\sin(2t)}{2}\right]_0^{\pi/2} = 0.$$

Lecture 20, November 16

- **Conservative vector fields.** We say that $\boldsymbol{F} = \langle F_1, F_2 \rangle$ is conservative, if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

In that case, $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$ for some function ϕ (the potential function) and

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla\phi \cdot d\boldsymbol{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

for any curve C from (x_0, y_0) to (x_1, y_1). Thus, the integral is path-independent.

....

Example 1. Take $\boldsymbol{F} = \langle 2xy, x^2 + 2y \rangle$. This vector field is conservative because

$$\frac{\partial F_1}{\partial y} = (2xy)_y = 2x, \qquad \frac{\partial F_2}{\partial x} = (x^2 + 2y)_x = 2x.$$

In particular, $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$ for some function ϕ and this means that

$$\phi_x = 2xy, \qquad \phi_y = x^2 + 2y.$$

To actually find the potential function ϕ, we note that integration gives

$$\begin{aligned} \phi &= \int 2xy\, dx = x^2 y + C_1(y), \\ \phi &= \int (x^2 + 2y)\, dy = x^2 y + y^2 + C_2(x) \end{aligned}$$

and then compare these two equations to get the potential function $\phi = x^2 y + y^2$.

Example 2. Let $\boldsymbol{F} = \langle 2xy, x^2 + 2y \rangle$ as before and consider the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r},$$

where C is the straight line from $(1, 0)$ to $(0, 1)$. Then we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{(1,0)}^{(0,1)} \nabla\phi \cdot d\boldsymbol{r} = \phi(0, 1) - \phi(1, 0) = 1.$$

Lecture 21, November 21

- **Green's theorem.** If R is a simply connected region in $\mathbb{R}^2$ whose boundary C is a simple, closed piecewise smooth curve oriented counterclockwise, then

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

In particular, the area of the region R may be computed using any of the formulas

$$\text{Area} = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

. . . .

Example 1. Consider the triangle C whose vertices are $(0,0)$, $(1,0)$ and $(1,2)$. Then

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_R (2xy^3 - x)\,dA,$$

where R is the interior of the triangle. This actually gives

$$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx = \int_0^1 \left[\frac{2xy^4}{4} - xy\right]_{y=0}^{2x} dx \\ &= \int_0^1 (8x^5 - 2x^2)\,dx = \frac{8}{6} - \frac{2}{3} = \frac{2}{3}.\end{aligned}$$

Example 2. Let C be the circle of radius 2 around the origin and let

$$\boldsymbol{F}(x,y) = \left\langle e^x - y^3, \cos y + x^3\right\rangle.$$

According to Green's theorem, we then have

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R (3x^2 + 3y^2)\,dA,$$

where R is the interior of the circle. Switching to polar coordinates, we find that

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi}\int_0^2 3r^3\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{3r^4}{4}\right]_{r=0}^2 d\theta = \int_0^{2\pi} 12\,d\theta = 24\pi.\end{aligned}$$

Lecture 22, November 23

- **Surface integrals.** The integral of $f(x, y, z)$ over a surface σ in $\mathbb{R}^3$ is

$$\iint_\sigma f(x, y, z)\, dS = \iint f(x(u, v), y(u, v), z(u, v)) \cdot ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, du\, dv,$$

where $\boldsymbol{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is the parametric equation of the surface.

- When the surface is the graph of $z = f(x, y)$, one has $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$.

. . . .

Example 1 (Cylinder). The parametric equation of the cylinder $x^2 + y^2 = 1$ is

$$\boldsymbol{r} = \langle x, y, z\rangle = \langle \cos\theta, \sin\theta, z\rangle$$

and it is obtained using cylindrical coordinates. In this case, we have

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_z &= \langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0, 0, 1\rangle = \langle \cos\theta, \sin\theta, 0\rangle\,, \\ ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| &= \sqrt{\cos^2\theta + \sin^2\theta} = 1 \end{aligned}$$

and one can use these facts to compute any surface integral over the cylinder.

Example 2 (Cone). The parametric equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$x^2 + y^2 = z^2 \quad\Longrightarrow\quad \boldsymbol{r} = \langle x, y, z\rangle = \langle z\cos\theta, z\sin\theta, z\rangle\,.$$

To compute a surface integral over the cone, one needs to compute

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_z &= \langle -z\sin\theta, z\cos\theta, 0\rangle \times \langle \cos\theta, \sin\theta, 1\rangle = \langle z\cos\theta, z\sin\theta, -z\rangle\,, \\ ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| &= \sqrt{z^2\cos^2\theta + z^2\sin^2\theta + z^2} = z\sqrt{2}. \end{aligned}$$

Example 3 (Sphere). The parametric equation of the sphere $x^2 + y^2 + z^2 = 1$ is

$$\boldsymbol{r} = \langle x, y, z\rangle = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

and it is obtained using spherical coordinates. In this case, we have

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_\phi &= \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle \times \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle \\ &= \left\langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\right\rangle = -(\sin\phi)\boldsymbol{r} \end{aligned}$$

and the fact that $||\boldsymbol{r}|| = 1$ implies that $||\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi|| = \sin\phi$.

Example 4 (A general example). The graph of $z = f(x, y)$ can be described by

$$\boldsymbol{r} = \langle x, y, z \rangle = \langle x, y, f(x, y) \rangle .$$

To compute a surface integral over this graph, one needs to compute

$$\begin{aligned} \boldsymbol{r}_x \times \boldsymbol{r}_y &= \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle , \\ ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| &= \sqrt{1 + f_x^2 + f_y^2}. \end{aligned}$$

Example 5. We compute the integral $\iint_\sigma z^2 \, dS$ in the case that σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between $z = 0$ and $z = 1$. As in Example 2, we have

$$\boldsymbol{r} = \langle z \cos\theta, z \sin\theta, z \rangle , \qquad ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| = z\sqrt{2}.$$

This implies $dS = z\sqrt{2} \, dz \, d\theta$, so the given integral becomes

$$\iint_\sigma z^2 \, dS = \int_0^{2\pi} \int_0^1 z^3 \sqrt{2} \, dz \, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{4} \, d\theta = \frac{\pi\sqrt{2}}{2}.$$

Example 6. Consider the lamina that occupies the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$. If its density is given by $\delta(x, y, z)$, then its mass is

$$\text{Mass} = \iint_\sigma \delta(x, y, z) \, dS.$$

Assume that δ is constant for simplicity. Since $z = f(x, y) = x^2 + y^2$, we have

$$||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4x^2 + 4y^2}$$

by Example 4. Using this fact and switching to polar coordinates, we find that

$$\begin{aligned} \text{Mass} &= \int \int \delta \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy \\ &= \delta \int_0^{2\pi} \int_0^1 (1 + 4r^2)^{1/2} \, r \, dr \, d\theta \\ &= \frac{\delta}{8} \int_0^{2\pi} \int_1^5 u^{1/2} \, du \, d\theta \qquad \boxed{u = 1 + 4r^2} \\ &= \frac{\delta}{8} \int_0^{2\pi} \frac{5^{3/2} - 1^{3/2}}{3/2} \, d\theta \\ &= \frac{\delta\pi}{6} (5\sqrt{5} - 1). \end{aligned}$$

Lecture 23, November 26

- **Flux.** The flux of the vector field $\boldsymbol{F}(x,y,z)$ through a surface σ in $\mathbb{R}^3$ is

$$\text{Flux} = \iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where $\boldsymbol{n}$ is the unit normal vector depending on the orientation of the surface. If σ is the graph of $z=f(x,y)$ oriented upwards, then $\boldsymbol{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$.

. . . .

Example 1. Let $\boldsymbol{F}=\langle x,y,z\rangle$ and let σ be the part of the paraboloid $z=1-x^2-y^2$ that lies above the xy-plane, oriented upwards. In this case, we have

$$f(x,y)=1-x^2-y^2 \quad\Longrightarrow\quad \boldsymbol{n}\,dS=\langle 2x,2y,1\rangle\;dx\,dy.$$

Taking the dot product with $\boldsymbol{F}=\langle x,y,1-x^2-y^2\rangle$, we end up with

$$\begin{aligned}\text{Flux} &= \iint (2x^2+2y^2+1-x^2-y^2)\,dx\,dy\\ &= \iint (x^2+y^2+1)\,dx\,dy\end{aligned}$$

and the projection of σ onto the xy-plane is the interior of the circle $x^2+y^2=1$, so

$$\begin{aligned}\text{Flux} &= \int_0^{2\pi}\int_0^1 (r^2+1)\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^3+r)\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_{r=0}^1 d\theta = \int_0^{2\pi}\frac{3}{4}\,d\theta = \frac{3\pi}{2}.\end{aligned}$$

Example 2. Let $\boldsymbol{F}=\langle 1,y,0\rangle$ and let σ be the part of the plane $x+y+z=1$ that lies in the first octant, oriented upwards. Then $z=f(x,y)=1-x-y$ and

$$\boldsymbol{n}\,dS=\langle -f_x,-f_y,1\rangle\;dx\,dy=\langle 1,1,1\rangle\;dx\,dy.$$

Taking the dot product with $\boldsymbol{F}=\langle 1,y,0\rangle$, we conclude that

$$\begin{aligned}\text{Flux} &= \iint (1+y)\,dx\,dy = \int_0^1\int_0^{1-y}(1+y)\,dx\,dy\\ &= \int_0^1 (1+y)(1-y)\,dy = \int_0^1 (1-y^2)\,dy = 1-\frac{1}{3}=\frac{2}{3}.\end{aligned}$$

Lecture 24, November 28

- **Divergence theorem.** The outward flux of $\boldsymbol{F}$ through a closed surface σ in $\mathbb{R}^3$ is

$$\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_G (\operatorname{div}\boldsymbol{F})\,dV,$$

where $\boldsymbol{n}$ is the outward unit normal vector and G is the solid enclosed by σ.

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Example 1. Let $\boldsymbol{F} = \langle 2x, 3y, z^2\rangle$ and let σ be the surface consisting of the six faces of the unit cube. Then the outward flux of $\boldsymbol{F}$ through σ is given by

$$\begin{aligned}\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS &= \iiint_G (2+3+2z)\,dV = \int_0^1\int_0^1\int_0^1 (5+2z)\,dx\,dy\,dz\\ &= \int_0^1\int_0^1 (5+2z)\,dy\,dz = \int_0^1 (5+2z)\,dz = 5+1 = 6.\end{aligned}$$

Example 2. Let $\boldsymbol{F} = \langle x^3, y^3, z^2\rangle$ and let σ be the surface of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=1$ (including the top and bottom parts). Then the outward flux through this surface may be computed as

$$\begin{aligned}\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS &= \iiint_G (3x^2+3y^2+2z)\,dV = \int_0^1\int_0^{2\pi}\int_0^2 (3r^3+2rz)\,dr\,d\theta\,dz\\ &= \int_0^1\int_0^{2\pi}\left[\frac{3r^4}{4}+r^2z\right]_{r=0}^2 d\theta\,dz = \int_0^1\int_0^{2\pi}(12+4z)\,d\theta\,dz\\ &= \int_0^1 2\pi(12+4z)\,dz = 2\pi(12+2) = 28\pi.\end{aligned}$$

Example 3. Let $\boldsymbol{F} = \langle y, x, z\rangle$ and let σ be the sphere $x^2+y^2+z^2=a^2$ of radius a around the origin. Then the outward flux through σ is given by

$$\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_G (0+0+1)\,dV = \iiint_G dV = \text{volume of } G = \frac{4\pi a^3}{3}.$$

Lecture 25, November 30

- **Stokes' theorem.** If σ is an oriented surface that is bounded by the curve C and C is positively oriented (according to the right hand rule), then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS.$$

And if σ is the graph of $z = f(x, y)$ oriented upwards, then $\boldsymbol{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy$.

. . . .

Example 1. Let $\boldsymbol{F} = \langle z, x, y\rangle$ and let σ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upwards. In this case, we have

$$z = f(x, y) = 1 - x^2 - y^2 \quad \Longrightarrow \quad \boldsymbol{n}\, dS = \langle 2x, 2y, 1\rangle\, dx\, dy$$

and one can easily check that

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \boldsymbol{k} = \langle 1, 1, 1\rangle .$$

Taking the dot product of these two vectors, we conclude that

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \iint_\sigma (2x + 2y + 1)\, dx\, dy = \int_0^{2\pi} \int_0^1 (2r^2 \cos\theta + 2r^2 \sin\theta + r)\, dr\, d\theta \\ &= \int_0^{2\pi} \left(\frac{2\cos\theta}{3} + \frac{2\sin\theta}{3} + \frac{1}{2}\right) d\theta = \left[\frac{2\sin\theta}{3} - \frac{2\cos\theta}{3} + \frac{\theta}{2}\right]_0^{2\pi} = \pi.\end{aligned}$$

Example 2. Let $\boldsymbol{F} = \langle z, x, y\rangle$ and let σ be the part of the plane $x + 2y + z = 4$ that lies in the first octant, oriented upwards. Arguing as before, we get

$$z = f(x, y) = 4 - x - 2y \quad \Longrightarrow \quad \boldsymbol{n}\, dS = \langle 1, 2, 1\rangle\, dx\, dy$$

as well as $\operatorname{curl} \boldsymbol{F} = \langle 1, 1, 1\rangle$, so Stokes' theorem implies that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (1 + 2 + 1)\, dx\, dy = 4 \iint_\sigma dx\, dy.$$

The values of x, y are determined by the projection onto the xy-plane. This is formed by the line $x + 2y = 4$ (that we get when $z = 0$) and the coordinate axes, hence

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 4 \int_0^2 \int_0^{4-2y} dx\, dy &= 4 \int_0^2 (4 - 2y)\, dy \\ &= 4\Big[4y - y^2\Big]_0^2 = 16.\end{aligned}$$

Lecture 26, December 3

- **Laplace transform.** Given a function $f(t)$, we define $\mathscr{L}(f)$ by the formula

$$\mathscr{L}(f) = \int_0^\infty e^{-st} f(t)\, dt, \qquad s > 0.$$

Some of the properties of the Laplace transform are listed in the following table.

Function	Laplace transform
1	$1/s$
e^{kt}	$1/(s-k)$
$f'(t)$	$s\mathscr{L}(f) - f(0)$

In addition, $\mathscr{L}(f+g) = \mathscr{L}(f) + \mathscr{L}(g)$ and $\mathscr{L}(cf) = c\mathscr{L}(f)$ for each constant c.

. . . .

Example 1. The Laplace transform of the function $y(t) = 2e^t + 3e^{2t}$ is given by

$$\mathscr{L}(y) = 2\mathscr{L}(e^t) + 3\mathscr{L}(e^{2t}) = \frac{2}{s-1} + \frac{3}{s-2} = \frac{5s-7}{(s-1)(s-2)}.$$

Example 2. We use the Laplace transform to solve $y'(t) = y(t)$ subject to $y(0) = y_0$. Taking the Laplace transform of both sides, we find that

$$\mathscr{L}(y') = \mathscr{L}(y) \quad \Longrightarrow \quad s\mathscr{L}(y) - y_0 = \mathscr{L}(y).$$

Next, we solve for $\mathscr{L}(y)$ and consult the table to conclude that

$$(s-1)\mathscr{L}(y) = y_0 \quad \Longrightarrow \quad \mathscr{L}(y) = \frac{y_0}{s-1} \quad \Longrightarrow \quad y(t) = y_0 e^t.$$

Example 3. We use the Laplace transform to solve $y'(t) - 2y(t) = 4$ subject to the initial condition $y(0) = 1$. Taking the Laplace transform of both sides gives

$$\begin{aligned} \mathscr{L}(y') - 2\mathscr{L}(y) = \mathscr{L}(4) \quad &\Longrightarrow \quad s\mathscr{L}(y) - y(0) - 2\mathscr{L}(y) = \frac{4}{s} \\ &\Longrightarrow \quad (s-2)\mathscr{L}(y) = y(0) + \frac{4}{s} = 1 + \frac{4}{s}. \end{aligned}$$

To use the table in this case, one needs to employ partial fractions to write

$$\mathscr{L}(y) = \frac{1}{s-2} + \frac{4}{s(s-2)} = \frac{1}{s-2} + \frac{2}{s-2} - \frac{2}{s} = \frac{3}{s-2} - \frac{2}{s}$$

and this is easily seen to imply that $y(t) = 3e^{2t} - 2$.

Lecture 27, December 5

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
1	$1/s$	e^{kt}	$1/(s-k)$
$y'(t)$	$s\mathscr{L}(y) - y(0)$	$\sin(kt)$	$k/(s^2+k^2)$
$y''(t)$	$s^2\mathscr{L}(y) - sy(0) - y'(0)$	$\cos(kt)$	$s/(s^2+k^2)$

. . . .

Example 1. We use the table above to solve the initial value problem

$$y''(t) + 4y(t) = 2e^t, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Taking the Laplace transform of both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + 4\mathscr{L}(y) = \frac{2}{s-1}$$

and we can solve for $\mathscr{L}(y)$ to find that

$$(s^2+4)\mathscr{L}(y) = s + \frac{2}{s-1} \quad \Longrightarrow \quad \mathscr{L}(y) = \frac{s}{s^2+4} + \frac{2}{(s-1)(s^2+4)}.$$

To handle the rightmost term, we have to decompose it into partial fractions as

$$\frac{2}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}.$$

Let us now determine the coefficients A, B, C. Clearing denominators gives

$$2 = A(s^2+4) + (Bs+C)(s-1)$$

and this identity should hold for all s. When $s = 1$, the identity reduces to

$$2 = 5A \quad \Longrightarrow \quad A = 2/5.$$

When $s = 0$, we get $2 = 4A - C$ and so $C = 4A - 2 = -2/5$. When $s = -1$, we get

$$2 = 5A - 2(C - B) = 2 + 4/5 + 2B \quad \Longrightarrow \quad B = 1 - 1 - 2/5 = -2/5.$$

We now employ this partial fractions decomposition to write

$$\mathscr{L}(y) = \frac{s}{s^2+4} + \frac{2/5}{s-1} - \frac{2s/5}{s^2+4} - \frac{2/5}{s^2+4}.$$

Consulting the table once again, we conclude that

$$\begin{aligned} y(t) &= \cos(2t) + \frac{2e^t}{5} - \frac{2\cos(2t)}{5} - \frac{\sin(2t)}{5} \\ &= \frac{3\cos(2t)}{5} + \frac{2e^t}{5} - \frac{\sin(2t)}{5}. \end{aligned}$$

Lecture 28, December 7

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s)$	t^n	$n!/s^{n+1}$
e^{kt}	$1/(s-k)$	$\sin(kt)$	$k/(s^2+k^2)$
$e^{kt}f(t)$	$F(s-k)$	$u(t-k)f(t-k)$	$e^{-ks}F(s)$

. . . .

Example 1. We compute $\mathscr{L}(t^2e^{4t})$. Ignoring the exponential factor, we find that

$$\mathscr{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}.$$

If we now include the exponential factor e^{4t}, then s becomes $s-4$ and we get

$$\mathscr{L}(t^2e^{4t}) = \frac{2}{(s-4)^3}.$$

Example 2. We compute $\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s^2-5s+6}\right)$. First, we use partial fractions to write

$$\frac{1}{s^2-5s+6} = \frac{1}{(s-3)(s-2)} = \frac{1}{s-3} - \frac{1}{s-2}$$

and we consult our table to find that

$$\mathscr{L}^{-1}\left(\frac{1}{s^2-5s+6}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-3}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}.$$

If we now include the exponential factor e^{-2s}, then t becomes $t-2$ and we get

$$\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s^2-5s+6}\right) = u(t-2)\cdot(e^{3t-6} - e^{2t-4}).$$

Example 3. We compute $\mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}\right)$. Since the denominator does not factor, we cannot use partial fractions in this case. Let us then complete the square to express

$$\frac{1}{s^2-4s+5} = \frac{1}{s^2-4s+4+1} = \frac{1}{(s-2)^2+1}$$

as a shifted version of $\frac{1}{s^2+1}$. According to our table, this implies

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \quad\Longrightarrow\quad \mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}\right) = e^{2t}\sin t.$$

Lecture 29, December 10

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s)$	t^n	$n!/s^{n+1}$
e^{kt}	$1/(s-k)$	$\sin(kt)$	$k/(s^2+k^2)$
$e^{kt}f(t)$	$F(s-k)$	$u(t-k)f(t-k)$	$e^{-ks}F(s)$
$\delta(t-k)$	e^{-ks}	$u(t-k)$	e^{-ks}/s

. . . .

Example 1. We use the table above to solve the initial value problem

$$y''(t) + y(t) = \delta(t-1) + u(t-2), \qquad y(0) = 0, \qquad y'(0) = 1.$$

Taking the Laplace transform of both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + \mathscr{L}(y) = e^{-s} + \frac{e^{-2s}}{s}$$

and we can solve for $\mathscr{L}(y)$ to find that

$$(s^2+1)\mathscr{L}(y) = 1 + e^{-s} + \frac{e^{-2s}}{s} \quad \Longrightarrow \quad \mathscr{L}(y) = \frac{1}{s^2+1} + \frac{e^{-s}}{s^2+1} + \frac{e^{-2s}}{s(s^2+1)}.$$

To handle the rightmost term, we have to decompose it into partial fractions as

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Let us now determine the coefficients A, B, C. Clearing denominators gives

$$1 = A(s^2+1) + (Bs+C)s = As^2 + A + Bs^2 + Cs$$

and we may compare coefficients of s to find that

$$A = 1, \qquad C = 0, \qquad B = -A = -1.$$

This gives rise to the partial fractions decomposition

$$\mathscr{L}(y) = \frac{1}{s^2+1} + \frac{e^{-s}}{s^2+1} + \frac{e^{-2s}}{s} - \frac{se^{-2s}}{s^2+1}.$$

Consulting the table once again, we conclude that

$$y(t) = \sin t + u(t-1)\sin(t-1) + u(t-2) - u(t-2)\cos(t-2).$$