Gradient vector

- The gradient of $f(x, y, z)$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$. This gives the direction of most rapid increase at each point and the rate of change in that direction is $||\nabla f||$.
- The direction of most rapid decrease is given by $-\nabla f$ and the rate of change in that direction is $-||\nabla f||$.

Tangent plane and normal line

- The tangent plane to the graph of $z = f(x, y)$ at the point (x_0, y_0, z_0) is the plane

$$z - f(x_0, y_0) = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0).$$

- The normal line to the graph of $z = f(x, y)$ at the point (x_0, y_0, z_0) has direction

$$\boldsymbol{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle .$$

Flux and surface integrals

- The flux of the vector field $\boldsymbol{F}(x, y, z)$ through a surface σ in $\mathbb{R}^3$ is given by

$$\text{Flux} = \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n} \, dS,$$

  where $\boldsymbol{n}$ is the unit normal vector depending on the orientation of the surface. If σ is the graph of $z = f(x, y)$ oriented upwards, then $\boldsymbol{n} \, dS = \langle -f_x, -f_y, 1 \rangle \, dx \, dy$.

- The surface integral of a function $H(x, y, z)$ over the graph of $z = f(x, y)$ is given by

$$\iint_\sigma H(x, y, z) \, dS = \iint_R H(x, y, f(x, y)) \cdot \sqrt{1 + f_x^2 + f_y^2} \; dx \, dy,$$

  where σ denotes the graph of $z = f(x, y)$ and R is its projection onto the xy-plane.

Change of variables

- **Cylindrical coordinates.** These are defined by the formulas

$$x = r \cos\theta, \qquad y = r \sin\theta, \qquad x^2 + y^2 = r^2, \qquad dV = r \, dz \, dr \, d\theta.$$

- **Spherical coordinates.** These are defined by the formulas

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi, \qquad dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

- **Formula for change of variables.** When it comes to double integrals, one has

$$\iint f(x,y)\,dx\,dy = \iint f(x(u,v),y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

  Here, the additional factor inside the integral is the absolute value of the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Mass and volume

- A two-dimensional lamina R with density $\delta(x,y)$ has mass $\iint_R \delta(x,y)\,dA$.
- A three-dimensional solid G with density $\delta(x,y,z)$ has mass $\iiint_G \delta(x,y,z)\,dV$.
- A three-dimensional solid G has volume $\iiint_G dV$.
- Let G be a solid which is bounded by $z=f(x,y)$ from above and by $z=g(x,y)$ from below. If its projection onto the xy-plane is the region R, then its volume is

$$\text{Volume} = \iiint_G dV = \iint_R [f(x,y)-g(x,y)]\,dA.$$

Divergence and curl

- The divergence of the vector field $\boldsymbol{F}(x,y,z) = \langle F_1, F_2, F_3\rangle$ is defined by

$$\operatorname{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- The curl of the vector field $\boldsymbol{F}(x,y,z) = \langle F_1, F_2, F_3\rangle$ is defined by

$$\operatorname{curl}\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.$$

Line integrals

- The integral of the function $f(x,y)$ over a curve C in the xy-plane is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot ||\boldsymbol{r}'(t)||\,dt,$$

  where $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ is the equation of the curve and $a \leq t \leq b$. The integrals

$$\int_C f(x,y)\,dx, \qquad \int_C f(x,y)\,dy, \qquad \int_C \boldsymbol{F}\cdot d\boldsymbol{r}$$

  are defined similarly in terms of $dx = x'(t)\,dt$, $dy = y'(t)\,dt$ and $d\boldsymbol{r} = \boldsymbol{r}'(t)\,dt$.

Conservative vector fields

- We say that the vector field $\boldsymbol{F}(x, y) = \langle F_1, F_2 \rangle$ is conservative, if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.
- Such fields have the form $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$ for some potential function ϕ.
- If C is a curve from (x_0, y_0) to (x_1, y_1) and $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$, then

  $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(x_1, y_1) - \phi(x_0, y_0).$$

  In particular, the line integral of a conservative vector field is path-independent.

Divergence, Green's and Stokes' theorems

- **Divergence theorem.** The outward flux of $\boldsymbol{F}$ through a closed surface σ in $\mathbb{R}^3$ is

  $$\iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_G (\operatorname{div} \boldsymbol{F})\, dV,$$

  where $\boldsymbol{n}$ is the outward unit normal vector and G is the solid enclosed by σ.
- **Green's theorem.** If R is a simply connected region in $\mathbb{R}^2$ whose boundary C is a simple, closed piecewise smooth curve oriented counterclockwise, then

  $$\oint_C F_1\, dx + F_2\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

- **Stokes' theorem.** If σ is an oriented surface that is bounded by the curve C and C is positively oriented (according to the right hand rule), then

  $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS.$$

  And if σ is the graph of $z = f(x, y)$ oriented upwards, then $\boldsymbol{n}\, dS = \langle -f_x, -f_y, 1 \rangle\, dx\, dy$.

Laplace transform

- Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s)$	1	$1/s$
t^n	$n!/s^{n+1}$	$\sin(kt)$	$k/(s^2 + k^2)$
e^{kt}	$1/(s - k)$	$\cos(kt)$	$s/(s^2 + k^2)$
$e^{kt} f(t)$	$F(s - k)$	$u(t - k) f(t - k)$	$e^{-ks} F(s)$
$\delta(t - k)$	e^{-ks}	$y'(t)$	$s\mathscr{L}(y) - y(0)$
$u(t - k)$	e^{-ks}/s	$y''(t)$	$s^2\mathscr{L}(y) - sy(0) - y'(0)$