2012 final exam solutions

1a. Applying the Laplace transform to both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + 9\mathscr{L}(y) = 6e^{-2\pi s} - \frac{9e^{-\pi s}}{s}$$

and we can rearrange terms to write this equation as

$$(s^2+9)\mathscr{L}(y) = s + 6e^{-2\pi s} - \frac{9e^{-\pi s}}{s}.$$

Next, we divide by s^2+9 and use partial fractions to find that

$$\begin{aligned}\mathscr{L}(y) &= \frac{s}{s^2+9} + \frac{6e^{-2\pi s}}{s^2+9} - \frac{9e^{-\pi s}}{s(s^2+9)} \\ &= \frac{s}{s^2+9} + \frac{6e^{-2\pi s}}{s^2+9} + \frac{se^{-\pi s}}{s^2+9} - \frac{e^{-\pi s}}{s}.\end{aligned}$$

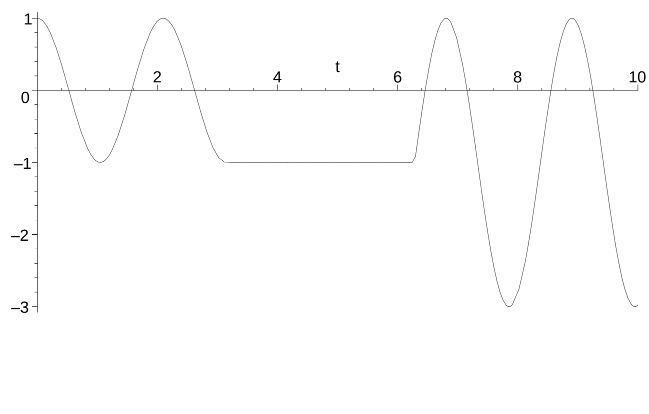
Using this fact and our table of Laplace transforms, we conclude that

$$\begin{aligned}y(t) &= \cos(3t) + 2u(t-2\pi)\sin(3t-6\pi) + u(t-\pi)\cos(3t-3\pi) - u(t-\pi) \\ &= \cos(3t) + 2u(t-2\pi)\sin(3t) - u(t-\pi)\cos(3t) - u(t-\pi).\end{aligned}$$

1b. The input function refers to the right hand side of the given equation. In this case, it is 0 when $t < \pi$, it is -9 when $\pi \leq t \neq 2\pi$ and it is plus infinity when $t = 2\pi$. As for the solution we found in part (a), this can be written in the form

$$y(t) = \left\{\begin{array}{cl} \cos(3t) & \text{if } t < \pi \\ -1 & \text{if } \pi \leq t < 2\pi \\ 2\sin(3t) - 1 & \text{if } t \geq 2\pi \end{array}\right\}.$$

A sketch of the graph of this function appears in the figure below.



2a. The direction of most rapid increase is given by the gradient $\nabla f = \langle f_x, f_y, f_z \rangle$, where

$$\begin{aligned} f_x &= \frac{(z^2 + x - y + 2\cos(3y - 2x))_x}{2\sqrt{z^2 + x - y + 2\cos(3y - 2x)}} \\ &= \frac{1 - 2\sin(3y - 2x) \cdot (-2)}{2\sqrt{z^2 + x - y + 2\cos(3y - 2x)}} = \frac{1 - 2\sin 0 \cdot (-2)}{2\sqrt{1 + 3 - 2 + 2\cos 0}} = \frac{1}{4} \end{aligned}$$

at the given point, while a similar computation gives $f_y = -1/4$ and $f_z = -1/2$. To find a unit vector $\boldsymbol{u}$ in the direction of the gradient, we simply divide by its length:

$$\begin{aligned} \nabla f = \left\langle \frac{1}{4}, -\frac{1}{4}, -\frac{1}{2} \right\rangle \quad &\Longrightarrow \quad ||\nabla f|| = \sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{4}} = \frac{\sqrt{6}}{4} \\ &\Longrightarrow \quad \boldsymbol{u} = \frac{\nabla f}{||\nabla f||} = \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle. \end{aligned}$$

2b. The projection of a vector onto the xz-plane is obtained by ignoring its y-coordinate. Thus, the projection of $\boldsymbol{u} = \langle 1/\sqrt{6}, -1/\sqrt{6}, -2/\sqrt{6} \rangle$ is $\langle 1/\sqrt{6}, -2/\sqrt{6} \rangle$.

2c. Since $\boldsymbol{u}$ is a unit vector in the direction of most rapid increase, $-\boldsymbol{u}$ is a unit vector in the direction of most rapid decrease.

2d. The projection of a vector onto the xy-plane is obtained by ignoring its z-coordinate. Thus, the projection of $-\boldsymbol{u} = \langle -1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6} \rangle$ is $\langle -1/\sqrt{6}, 1/\sqrt{6} \rangle$.

2e. The rate of change in the direction of $\pm\boldsymbol{u}$ is equal to $\pm||\nabla f|| = \pm\frac{1}{4}\sqrt{6}$, respectively.

3a. Let us first simplify the given equation and write

$$z = f(x, y) = \frac{1}{3}\ln\Big(3\cos(2x - y) + 6x^2 - 6xy^2 - y^3 + 31\Big) - \ln 2.$$

Differentiating with respect to x, we then find that

$$\begin{aligned} f_x &= \frac{1}{3} \cdot \frac{-3\sin(2x - y) \cdot 2 + 12x - 6y^2}{3\cos(2x - y) + 6x^2 - 6xy^2 - y^3 + 31} \\ &= \frac{1}{3} \cdot \frac{12 - 24}{8} = -\frac{1}{2} \end{aligned}$$

at the given point, while a similar computation gives $f_y = -3/2$. Noting that

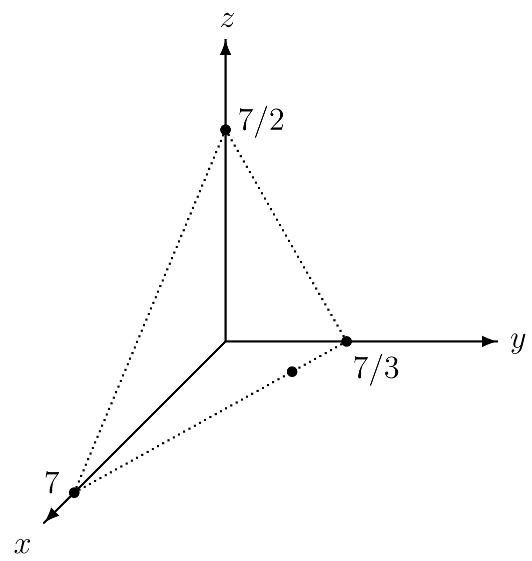
$$\begin{aligned} z_0 = f(1, 2) &= \frac{1}{3}\ln\Big(3\cos 0 + 6 - 24 - 8 + 31\Big) - \ln 2 \\ &= \frac{1}{3}\ln 2^3 - \ln 2 = 0, \end{aligned}$$

we conclude that the equation of the tangent plane at the given point is

$$z = f_x(1, 2) \cdot (x - 1) + f_y(1, 2) \cdot (y - 2) = -\frac{x + 3y - 7}{2}.$$

3b. The plane intersects the x-axis when $y = z = 0$, in which case $x - 7 = 0$. This gives the point $(7, 0, 0)$, while the points $(0, 7/3, 0)$ and $(0, 0, 7/2)$ can be found similarly.

3c. A graph of the tangent plane appears in the figure below. The point $P(1, 2, 0)$ lies on both the xy-plane and the tangent plane, so it lies along the dotted line.



3d. Write the equation of the tangent plane in the form

$$2z = -x - 3y + 7 \quad \Longrightarrow \quad x + 3y + 2z = 7.$$

The normal line passes through $(1, 2, 0)$ with direction $\langle 1, 3, 2 \rangle$, so its equation is

$$x = 1 + t, \qquad y = 2 + 3t, \qquad z = 2t.$$

3e. The normal line should be perpendicular to the tangent plane at the given point.

4a. The projection is the region that lies between the graphs of $y = x$ and $y = \sqrt{x}$.

4b. The volume of the solid is the double integral of the function $z = f(x, y)$, namely

$$\text{Volume} = \int_0^1 \int_x^{\sqrt{x}} \frac{e^{1-x} - \sin(\pi x/2)}{1 - x} \cdot y \, dy \, dx.$$

When it comes to the inner integral, one easily finds that

$$\int_x^{\sqrt{x}} y \, dy = \left[\frac{y^2}{2} \right]_{y=x}^{\sqrt{x}} = \frac{x - x^2}{2} = \frac{x(1-x)}{2}.$$

Using this fact and an integration by parts, we conclude that

$$\begin{aligned}
\text{Volume} &= \int_0^1 \left(e^{1-x} - \sin\frac{\pi x}{2}\right) \cdot \frac{x}{2}\,dx \\
&= \left[\left(-e^{1-x} + \frac{2}{\pi}\cdot\cos\frac{\pi x}{2}\right)\cdot\frac{x}{2}\right]_0^1 - \int_0^1 \left(-e^{1-x} + \frac{2}{\pi}\cdot\cos\frac{\pi x}{2}\right)\cdot\frac{1}{2}\,dx \\
&= \left[\left(-e^{1-x} + \frac{2}{\pi}\cdot\cos\frac{\pi x}{2}\right)\cdot\frac{x}{2} - \left(e^{1-x} + \frac{4}{\pi^2}\cdot\sin\frac{\pi x}{2}\right)\cdot\frac{1}{2}\right]_0^1 \\
&= \frac{e}{2} - \frac{2}{\pi^2} - 1.
\end{aligned}$$

Note: This integral was somewhat messy, so I am pretty sure that there was a typo in the statement of this problem and that the problem was meant to be easier.

4c. This is really a question about L'Hôpital's rule, which is not part of this course:

$$\lim_{x\to 1} \frac{e^{1-x} - \sin(\pi x/2)}{1-x} = \lim_{x\to 1} \frac{-e^{1-x} - \cos(\pi x/2)\cdot(\pi/2)}{-1} = 1.$$

5a. The given line integral is path-independent because

$$(6xy - 3\pi^3 \sin y)_x = 6y = (3y^2 + 2\pi^3 \cos x)_y.$$

5b. A potential function ϕ is a function that satisfies the equations

$$\phi_x = 3y^2 + 2\pi^3\cos x, \qquad \phi_y = 6xy - 3\pi^3 \sin y.$$

Integrating the first equation with respect to x gives

$$\phi = \int (3y^2 + 2\pi^3\cos x)\,dx = 3xy^2 + 2\pi^3 \sin x + C_1(y),$$

and we may similarly integrate the second equation to get

$$\phi = \int (6xy - 3\pi^3\sin y)\,dy = 3xy^2 + 3\pi^3\cos y + C_2(x).$$

Thus, one can always take $\phi(x,y) = 3xy^2 + 2\pi^3 \sin x + 3\pi^3\cos y$.

5c. According to the fundamental theorem, the value of the integral is

$$\phi(\pi, \pi/2) - \phi(-\pi/2, \pi) = 3\pi^3/4 - (-13\pi^3/2) = 29\pi^3/4.$$

5d. The first line segment is from $(-\pi/2, \pi)$ to (π, π) and its parametric equation is

$$\boldsymbol{r}(t) = \langle t, \pi \rangle, \qquad -\pi/2 \leq t \leq \pi.$$

This gives $x = t$ and also $y = \pi$, so we get

$$I_1 = \int_{-\pi/2}^{\pi} (3\pi^2 + 2\pi^3 \cos t)\, dt = \Big[3\pi^2 t + 2\pi^3 \sin t\Big]_{-\pi/2}^{\pi} = 13\pi^3/2.$$

The second line segment is from (π, π) to $(\pi, \pi/2)$ and its parametric equation is

$$\boldsymbol{r}(t) = \langle \pi, t \rangle, \qquad \pi \leq t \leq \pi/2$$

because the starting point occurs when $t = \pi$. This gives $x = \pi$ and $y = t$, so

$$I_2 = \int_{\pi}^{\pi/2} (6\pi t - 3\pi^3 \sin t)\, dt = \Big[3\pi t^2 + 3\pi^3 \cos t\Big]_{\pi}^{\pi/2} = 3\pi^3/4.$$

In particular, $I_1 + I_2 = 29\pi^3/4$ and this agrees with our answer in part (c).

6a. Cylindrical coordinates are defined by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

6b. The surface $x^2 + y^2 = a^2$ describes a cylinder of radius a around the z-axis, while the surface $z = a$ is a plane. The boundary of the solid which belongs to the xy-plane is the annulus that lies inside the circle $x^2 + y^2 = 4$ but outside the circle $x^2 + y^2 = 1$. To find the volume of the solid using cylindrical coordinates, we note that $1 \leq r \leq 2$ by above and that $0 \leq z \leq 3$ by assumption, so

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi} \int_1^2 \int_0^3 r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_1^2 3r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{3(2^2 - 1^2)}{2}\, d\theta = \frac{9}{2} \cdot 2\pi = 9\pi.\end{aligned}$$

To find the mass of the solid, we proceed similarly to get

$$\begin{aligned}\text{Mass} &= \int_0^{2\pi} \int_1^2 \int_0^3 re^{-r^2} e^{-z}\, dz\, dr\, d\theta = \int_0^{2\pi} \int_1^2 re^{-r^2}(1 - e^{-3})\, dr\, d\theta \\ &= (1 - e^{-3}) \int_0^{2\pi} \left[\frac{e^{-r^2}}{-2}\right]_1^2 d\theta = (1 - e^{-3}) \cdot \frac{(e^{-1} - e^{-4})}{2} \cdot 2\pi \\ &= (1 - e^{-3})^2 \cdot e^{-1}\pi.\end{aligned}$$