2011 final exam solutions

1a. Applying the Laplace transform to both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + 4\mathscr{L}(y) = \frac{8e^{-\pi s}}{s} - 8e^{-3\pi s}$$

and we can rearrange terms to write this equation as

$$(s^2+4)\mathscr{L}(y) = 2s + \frac{8e^{-\pi s}}{s} - 8e^{-3\pi s}.$$

Next, we divide by s^2+4 and use partial fractions to find that

$$\begin{aligned}\mathscr{L}(y) &= \frac{2s}{s^2+4} + \frac{8e^{-\pi s}}{s(s^2+4)} - \frac{8e^{-3\pi s}}{s^2+4} \\ &= \frac{2s}{s^2+4} + \frac{2e^{-\pi s}}{s} - \frac{2se^{-\pi s}}{s^2+4} - \frac{8e^{-3\pi s}}{s^2+4}.\end{aligned}$$

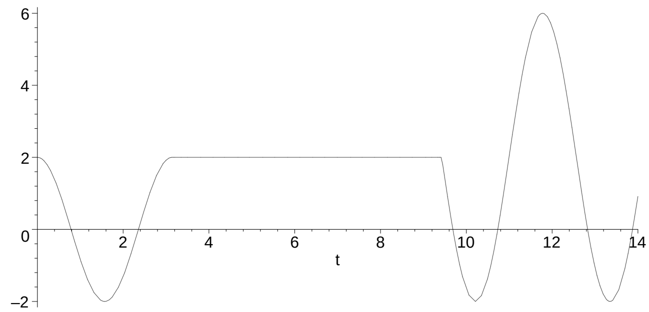
Using this fact and our table of Laplace transforms, we conclude that

$$\begin{aligned}y(t) &= 2\cos(2t) + 2u(t-\pi) - 2u(t-\pi)\cos(2t-2\pi) - 4u(t-3\pi)\sin(2t-6\pi) \\ &= 2\cos(2t) + 2u(t-\pi) - 2u(t-\pi)\cos(2t) - 4u(t-3\pi)\sin(2t).\end{aligned}$$

1b. The input function refers to the right hand side of the given equation. In this case, it is 0 when $t < \pi$, it is 8 when $\pi \leq t \neq 3\pi$ and it is minus infinity when $t = 3\pi$. As for the solution we found in part (a), this can be written in the form

$$y(t) = \left\{\begin{array}{cl} 2\cos(2t) & \text{if } t < \pi \\ 2 & \text{if } \pi \leq t < 3\pi \\ 2 - 4\sin(2t) & \text{if } t \geq 3\pi \end{array}\right\}.$$

A sketch of the graph of this function appears in the figure below.



2a. The direction of most rapid increase is given by the gradient $\nabla f = \langle f_x, f_y, f_z \rangle$, where

$$\begin{aligned} f_x &= \frac{(y^2 - \sin(3x - 2z))_x}{2\sqrt{y^2 - \sin(3x - 2z)}} \\ &= \frac{-\cos(3x - 2z) \cdot 3}{2\sqrt{y^2 - \sin(3x - 2z)}} = \frac{-\cos 0 \cdot 3}{2\sqrt{1 - \sin 0}} = -\frac{3}{2} \end{aligned}$$

at the given point, while a similar computation gives $f_y = -1$ and $f_z = 1$. To find a unit vector $\boldsymbol{u}$ in the direction of the gradient, we simply divide by its length:

$$\begin{aligned} \nabla f = \left\langle -\frac{3}{2}, -1, 1 \right\rangle \quad &\Longrightarrow \quad ||\nabla f|| = \sqrt{\frac{9}{4} + 1 + 1} = \frac{\sqrt{17}}{2} \\ &\Longrightarrow \quad \boldsymbol{u} = \frac{\nabla f}{||\nabla f||} = \left\langle -\frac{3}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}} \right\rangle. \end{aligned}$$

2b. The projection of a vector onto the xy-plane is obtained by ignoring its z-coordinate. Thus, the projection of $\boldsymbol{u} = \langle -3/\sqrt{17}, -2/\sqrt{17}, 2/\sqrt{17} \rangle$ is $\langle -3/\sqrt{17}, -2/\sqrt{17} \rangle$.

2c. Since $\boldsymbol{u}$ is a unit vector in the direction of most rapid increase, $-\boldsymbol{u}$ is a unit vector in the direction of most rapid decrease.

2d. The projection of a vector onto the yz-plane is obtained by ignoring its x-coordinate. Thus, the projection of $-\boldsymbol{u} = \langle 3/\sqrt{17}, 2/\sqrt{17}, -2/\sqrt{17} \rangle$ is $\langle 2/\sqrt{17}, -2/\sqrt{17} \rangle$.

2e. The rate of change in the direction of $\pm\boldsymbol{u}$ is equal to $\pm||\nabla f|| = \pm\frac{1}{2}\sqrt{17}$, respectively.

3a. Let us first simplify the given equation and write

$$z = f(x, y) = \frac{1}{3} \ln\left(2x^2 - 3y^3 - 3xy^2 + 21\right) - \ln 3.$$

Differentiating with respect to x, we then find that

$$f_x = \frac{1}{3} \cdot \frac{4x - 3y^2}{2x^2 - 3y^3 - 3xy^2 + 21} = \frac{1}{3} \cdot \frac{12 - 12}{27} = 0$$

at the given point, while a similar computation gives $f_y = 0$ as well. Noting that

$$\begin{aligned} z_0 = f(3, -2) &= \frac{1}{3} \ln\left(18 + 24 - 36 + 21\right) - \ln 3 \\ &= \frac{1}{3} \ln 3^3 - \ln 3 = 0, \end{aligned}$$

we conclude that the equation of the tangent plane at the given point is

$$z = f_x(3, -2) \cdot (x - 3) + f_y(3, -2) \cdot (y + 2) = 0.$$

3b. The tangent plane $z = 0$ is merely the xy-plane.

3c. The normal line passes through $(3, -2, 0)$ with direction $\langle 0, 0, 1\rangle$, so its equation is

$$x = 3, \qquad y = -2, \qquad z = t.$$

3d. The normal line has direction $\langle 0, 0, 1\rangle$, so it is parallel to the z-axis.

4a. The surface is a cylinder of radius $\sqrt{8}$.

4b. The surface lies above the rectangle R, so its projection onto the xy-plane is just R.

4c. Solving the given equation for z, one finds that

$$(z-1)^2 = 8 - (x-2)^2 \quad \Longrightarrow \quad z = 1 + \sqrt{8 - (x-2)^2}.$$

In particular, we have $z = f(x, y)$ for some function f and we need to integrate

$$dS = \sqrt{1 + f_x^2 + f_y^2}\ dx\, dy$$

over the region R. Since $f(x, y) = 1 + \sqrt{8 - (x-2)^2}$ by above, we get

$$f_x = -\frac{2(x-2)}{2\sqrt{8-(x-2)^2}} \quad \Longrightarrow \quad 1 + f_x^2 + f_y^2 = 1 + \frac{(x-2)^2}{8-(x-2)^2} = \frac{8}{8-(x-2)^2}$$

and so the area of the surface is

$$\text{Area} = \iint_R dS = \int_0^4 \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\sqrt{8}\, dy\, dx}{\sqrt{8-(x-2)^2}} = \int_0^4 \frac{2\sqrt{2}\sqrt{8}\, dx}{\sqrt{8-(x-2)^2}}.$$

To compute the rightmost integral, we use an analogue of polar coordinates:

$$\begin{aligned} 8 - (x-2)^2 = u^2 \quad &\Longrightarrow \quad (x-2)^2 + u^2 = 8 \\ &\Longrightarrow \quad x - 2 = \sqrt{8}\sin\theta, \qquad u = \sqrt{8}\cos\theta. \end{aligned}$$

In other words, we use the substitution $x = 2 + \sqrt{8}\sin\theta$. Note that $x = 0$ if and only if $\sin\theta = -1/\sqrt{2}$, while $x = 4$ if and only if $\sin\theta = 1/\sqrt{2}$. This actually gives

$$\text{Area} = \int_0^4 \frac{8\, dx}{\sqrt{8-(x-2)^2}} = \int_{-\pi/4}^{\pi/4} \frac{8 \cdot \sqrt{8}\cos\theta\, d\theta}{\sqrt{8}\cos\theta} = \int_{-\pi/4}^{\pi/4} 8\, d\theta = 4\pi.$$

5a. The given line integral is path-independent because

$$(4xy - 3y)_x = 4y = (x + 2y^2 - 3x^2)_y.$$

5b. A potential function ϕ is a function that satisfies the equations

$$\phi_x = x + 2y^2 - 3x^2, \qquad \phi_y = 4xy - 3y.$$

Integrating the first equation with respect to x gives

$$\phi = \int (x + 2y^2 - 3x^2)\,dx = \frac{x^2}{2} + 2xy^2 - x^3 + C_1(y),$$

and we may similarly integrate the second equation to get

$$\phi = \int (4xy - 3y)\,dy = 2xy^2 - \frac{3y^2}{2} + C_2(x).$$

Thus, one can always take $\phi(x,y) = 2xy^2 + \frac{1}{2}x^2 - x^3 - \frac{3}{2}y^2$.

5c. According to the fundamental theorem, the value of the integral is

$$\phi(1/2, 1) - \phi(-1, 0) = -1/2 - 3/2 = -2.$$

5d. The first line segment is from $(-1, 0)$ to $(1/2, 0)$ and its parametric equation is

$$\boldsymbol{r}(t) = \langle t, 0 \rangle, \qquad -1 \leq t \leq 1/2.$$

This gives $x = t$ and also $y = 0$, so we get

$$I_1 = \int_{-1}^{1/2} (t - 3t^2)\,dt = \left[\frac{t^2}{2} - t^3 \right]_{-1}^{1/2} = -3/2.$$

The second line segment is from $(1/2, 0)$ to $(1/2, 1)$ and its parametric equation is

$$\boldsymbol{r}(t) = \langle 1/2, t \rangle, \qquad 0 \leq t \leq 1.$$

This gives $x = 1/2$ and $y = t$, so we get

$$I_2 = \int_0^1 (2t - 3t)\,dt = -\int_0^1 t\,dt = -1/2.$$

In particular, $I_1 + I_2 = -2$ and this agrees with our answer in part (c).

6a. Spherical coordinates are defined by the equations

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

6b. The surface $x^2 + y^2 + z^2 = a^2$ describes a sphere of radius a, while the surface $z = 0$ is the xy-plane. The boundary of the solid which belongs to the xy-plane is the annulus that lies inside the circle $x^2 + y^2 = 4$ but outside the circle $x^2 + y^2 = 1$. To find the volume of the solid using spherical coordinates, we note that $1 \leq \rho \leq 2$ and that the points above the xy-plane are those with angles $0 \leq \phi \leq \pi/2$. This implies that

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{\rho^3}{3} \right]_1^2 \sin\phi \, d\phi \, d\theta \\
&= \frac{7}{3} \int_0^{2\pi} \Big[-\cos\phi \Big]_0^{\pi/2} d\theta = \frac{14\pi}{3}.
\end{aligned}$$

To find the mass of the solid, we proceed similarly to get

$$\begin{aligned}
\text{Mass} &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho e^{-\rho^2} \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{e^{-\rho^2}}{-2} \right]_1^2 \sin\phi \, d\phi \, d\theta \\
&= \frac{e^{-1} - e^{-4}}{2} \int_0^{2\pi} \Big[-\cos\phi \Big]_0^{\pi/2} d\theta = (e^{-1} - e^{-4})\pi.
\end{aligned}$$