UNIVERSITY OF DUBLIN

XMA2E011

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

SF Engineers
SF MSISS
SF MEMS

Trinity Term 2011

MODULE MA2E01 — ENGINEERING MATHEMATICS III

Thursday, May 5 | Exam Hall | 9.30 — 11.30

Dr. Sergey Frolov

ATTEMPT QUESTION 1 and FOUR OTHER QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (a) 14 marks. Solve the following initial value problem by the Laplace transform

$$y'' + 4\,y = 8\,u(t-\pi) - 8\,\delta(t-3\pi)\,, \qquad y(0) = 2\,, \quad y'(0) = 0\,.$$

(b) 4 marks. Sketch the input function and the solution.

Show the details of your work.

2. Consider the function

$$f(x,y,z) = \sqrt{y^2 - \sin(3x - 2z)}\,, \quad \text{and the point } P(2,-1,3)\,.$$

(a) 10 marks. Find a unit vector in the direction in which f increases most rapidly at the point P.

(b) 1 marks. Sketch the projection of the vector onto the xy-plane

(c) 3 marks. Find a unit vector in the direction in which f decreases most rapidly at the point P.

(d) 1 marks. Sketch the projection of the vector onto the yz-plane

(e) 3 marks. Find the rate of change of f at the point P in these directions.

Show the details of your work.

3. Consider the surface

$$z = \ln \frac{\sqrt[3]{2x^2 - 3y^3 - 3xy^2 + 21}}{3}$$

(a) 10 marks. Find an equation for the tangent plane to the surface at the point $P(3,-2,0)$.

(b) 1 marks. Sketch the tangent plane.

(c) 6 marks. Find parametric equations for the normal line to the surface at the point $P(3,-2,0)$.

(d) 1 marks. Sketch the normal line to the surface at the point $P(3,-2,0)$.

Show the details of your work.

4. Consider the portion of the surface $(x-2)^2+(z-1)^2-8=0$ that is above the rectangle

$$R=\{(x,y): 0\le x\le 4\,,\ -\sqrt{2}\le y\le \sqrt{2}\}.$$

(a) 3 marks. What is the surface?

(b) 2 marks. Sketch the projection of the portion onto the xy-plane.

(c) 13 marks. Use double integration to find the area of the portion.

Show the details of your work.

5. (a) 2 marks. Show that the integral below is independent of the path

$$\int_{(-1\,,0)}^{(1/2\,,1)} (-3x^2+x+2y^2)\,dx-(3y-4xy)\,dy\,.$$

(b) 10 marks. Find the potential function $\phi(x,y)$.

(c) 2 marks. Use the Fundamental Theorem of Line Integrals to find the value of the integral.

(d) Choose the integration path C between the points $(-1\,,0)$ and $(1/2\,,1)$ to be a curve formed from two line segments C_1 and C_2, where C_1 is joining $(-1\,,0)$ and $(1/2\,,0)$, and C_2 is joining $(1/2\,,0)$ and $(1/2\,,1)$.

i. 1 marks. Plot the integration path C, and show its orientation on the plot.

ii. 3 marks. Parameterize C_1 and C_2, and evaluate

$$\int_C (-3x^2+x+2y^2)\,dx-(3y-4xy)\,dy\,.$$

Show the details of your work.

6. (a) 3 marks. Express rectangular coordinates in terms of spherical coordinates

   (b) Consider the solid G bounded by the surfaces $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ and below by the surface $z=0$.

      i. 1 marks. What is the surface $x^2+y^2+z^2=1$?

      ii. 1 marks. What is the surface $x^2+y^2+z^2=4$?

      iii. 1 marks. What is the surface $z=0$?

      iv. 1 marks. Sketch the part of the boundary of the solid G which belongs to the surface $z=0$.

      v. 5 marks. Use triple integral and spherical coordinates to compute the volume V of the solid G.

      vi. 6 marks. Use triple integral and spherical coordinates to find the mass M of the solid G if its density is

$$\delta(x,y,z) = \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}\,.$$

Show the details of your work.

Useful Formulae

1. Let $\mathbf{r}(t)$ be a vector function with values in $\mathbf{R}^3$: $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$.

   (a) Its derivative is $\frac{d\mathbf{r}}{dt} = \left(\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right)$.

   (b) The magnitude of this vector is $||\frac{d\mathbf{r}}{dt}|| = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}$.

   (c) The unit tangent vector is $\mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{||\frac{d\mathbf{r}}{dt}||}$.

   (d) The vector equation of the line tangent to the graph of $\mathbf{r}(t)$ at the point $P = (x_0, y_0, z_0)$ corresponding to $t = t_0$ on the curve is $\mathbf{R}(t) = \mathbf{r}_0 + (t - t_0)\,\mathbf{v}_0$, where $\mathbf{r}_0 = \mathbf{r}(t_0)$ and $\mathbf{v}_0 = \frac{d\mathbf{r}}{dt}(t_0)$.

   (e) The arc length of the graph of $\mathbf{r}(t)$ between t_1 and t_2 is $L = \int_1^2 ||\frac{d\mathbf{r}}{dt}||\, dt$.

   (f) The arc length parameter s having $\mathbf{r}(t_0)$ as its reference point is $s = \int_{t_0}^{t} ||\frac{d\mathbf{r}}{du}||\, du$.

2. Let σ be a surface in $\mathbf{R}^3$: $z = f(x, y)$

   (a) The slope k_x of the surface in the x-direction at the point (x_0, y_0) is $k_x = \frac{\partial z}{\partial x}(x_0, y_0)$.

   (b) The slope k_y of the surface in the y-direction at the point (x_0, y_0) is $k_y = \frac{\partial z}{\partial y}(x_0, y_0)$.

   (c) The equation for the tangent plane to the surface at the point $P = (x_0, y_0, z_0)$ is $z = z_0 + k_x(x - x_0) + k_y(y - y_0)$.

   (d) Parametric equations for the normal line to the surface at $P = (x_0, y_0, z_0)$ are $\mathbf{r}(t) = \mathbf{r}_0 + t(-k_x\mathbf{i} - k_y\mathbf{j} + \mathbf{k})$, $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$.

   (e) The volume under the surface and over a region R in the xy-plane is $V = \iint_R f(x, y)\, dA$.

   (f) The area of the portion of the surface that is above a region R in the xy-plane is $S = \iint_\sigma dS = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

   (g) The mass of the lamina with the density $\delta(x, y, z)$ that is the portion of the surface that is above a region R in the xy-plane is $M = \iint_\sigma \delta(x, y, z)\, dS = \iint_R \delta(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

3. The local linear approximation of the function $z = f(x, y)$ at the point (x_0, y_0) is

   $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

4. Let $f(x, y, z)$ be a function of three variables

   (a) The gradient of f is $\nabla f = (f_x, f_y, f_z)$.

   (b) f increases most rapidly in the direction of its gradient, and the rate of change of f in this direction is equal to $||\nabla f||$.

   (c) If f is smooth then its critical points satisfy $f_x = f_y = f_z = 0$.

5. Let R be a region in the xy-plane bounded by the curves $y = g(x)$, $y = h(x)$, $x = a$, $x = b$, and $g \leq h$ for $a \leq x \leq b$. Then the double integral over the region is

   $\iint_R f(x, y)\, dA = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y) dy\right] dx$.

6. Let R be a region in the xy-plane bounded by the curves (in polar coordinates)

   $r = r_1(\theta)$, $r = r_2(\theta)$, $\theta = \alpha$, $\theta = \beta$ and $r_1 \leq r_2$ for $\alpha \leq \theta \leq \beta$. Then the double integral over the region is

   $\iint_R f(r, \theta)\, dA = \iint_R f(r, \theta)\, r dr\, d\theta = \int_\alpha^\beta \left[\int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r dr\right] d\theta$.

7. Let R be a plain lamina with density $\delta(x, y)$.

   (a) Its mass is equal to $M = \iint_R \delta(x, y)\, dA$.

   (b) The x-coordinate of its centre of gravity is equal to $x_{cg} = \frac{1}{M} \iint_R x\, \delta(x, y)\, dA$.

   (c) The y-coordinate of its centre of gravity is equal to $y_{cg} = \frac{1}{M} \iint_R y\, \delta(x, y)\, dA$.

8. Let G be a simple solid whose projection onto the xy-plane is a region R. G is bounded by a surface $z = g(x, y)$ from below and by a surface $z = h(x, y)$ from above.

   (a) The triple integral over the solid is $\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{g(x,y)}^{h(x,y)} f(x, y, z)\, dz\right] dA$.

   (b) The volume of the solid is $V = \iiint_G dV = \iint_R [h(x, y) - g(x, y)]\, dA$.

9. Let G be a solid enclosed between the two surfaces (in spherical coordinates)

   $r = g(\theta\,, \phi)$, $r = h(\theta\,, \phi)$.

(a) The triple integral over the solid is

$\iiint_G f(r,\theta,\phi)\,dV = \int_0^{2\pi}\left(\int_0^{\pi}\left[\int_{g(\theta,\phi)}^{h(\theta,\phi)} f(r,\theta,\phi)\,r^2dr\right]\sin\phi\,d\phi\right)d\theta\,.$

(b) The volume of the solid is $V = \iiint_G dV = \int_0^{2\pi}\left(\int_0^{\pi}\left[\int_{g(\theta,\phi)}^{h(\theta,\phi)} r^2dr\right]\sin\phi\,d\phi\right)d\theta\,.$

(c) The mass of the solid with the density $\delta(r,\theta,\phi)$ is $M = \iiint_G \delta(r,\theta,\phi)\,dV\,.$

10. Let a region R_{xy} in the xy-plane be mapped to a region R_{uv} in the uv-plane under the change of variables $u = u(x,y)\,,\ v = v(x,y)$.

 (a) The magnitude of the Jacobian of the change is $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|\,.$

 (b) The integral over R_{xy} is $\iint_{R_{xy}} f(x,y)\,dA_{xy} = \iint_{R_{uv}} f\left(x(u,v),y(u,v)\right)\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} dA_{uv}\,.$

11. The area of the surface that extends upward from the curve $x = x(t)\,,\ y = y(t)\,,\ a \le t \le b$ in the xy-plane to the surface $z = f(x,y)$ is given by the following line integral $A = \int_C z\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,.$

12. Consider a line integral $\int_C f(x,y)\,dx + g(x,y)\,dy$, and let $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ be the endpoints of the curve C.

 (a) The line integral is independent of the path if $\partial_y f(x,y) = \partial_x g(x,y)\,.$

 (b) Then there is a potential function $\phi(x,y)$ satisfying $\frac{\partial\phi}{\partial x} = f(x,y)\,,\ \frac{\partial\phi}{\partial y} = g(x,y)\,,$

 (c) and the Fundamental Theorem of Line Integrals says that

 $\int_C f(x,y)\,dx + g(x,y)\,dy = \int_P^Q \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy = \phi(x,y)|_P^Q = \phi(x_Q,y_Q) - \phi(x_P,y_P)\,.$

13. Let a closed curve C be oriented counterclockwise, and be the boundary of a simply connected region R in the xy-plane. By Green's Theorem we have

$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right)dA$

14. Let $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field.

 (a) If σ is the surface $z = f(x,y)$, oriented by upward unit normals $\mathbf{n}$, and R is the projection of σ onto the xy-plane then

 flux $= \iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(-M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right)dA\,.$

(b) If σ is the surface $z = f(x, y)$, oriented by downward unit normals $\mathbf{n}$, and R is the projection of σ onto the xy-plane then

flux $= \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \left(M\frac{\partial f}{\partial x} + N\frac{\partial f}{\partial y} - P \right) dA\,.$

(c) According to the Divergence Theorem the flux of $\mathbf{F}$ across a closed surface σ with outward orientation is

flux $= \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{F}\, dV\,, \quad \operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\,.$

(d) If σ is an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation then, according to Stokes' Theorem

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS\,,\ \operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}\,.$

15. The Laplace transform of a function $f(t)$ is the function $F(s)$ defined by

$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt\,, \quad f(t) = \mathcal{L}^{-1}(F(s))\,.$

Function	*Transform*	*Function*	*Transform*
e^{at}	$\frac{1}{s-a}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sinh \omega t$	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at} \cosh \omega t$	$\frac{s-a}{(s-a)^2-\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$	$\delta(t-a)$	e^{-as}

16. Let $F(s) = \mathcal{L}(f(t))$, then $\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$;

$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)\,;\ \mathcal{L}\left(tf(t)\right) = -\frac{dF(s)}{ds}\,;\ \mathcal{L}\left(f(kt)\right) = \frac{1}{k}F\left(\frac{s}{k}\right)\,.$

17. Let $Y(s) = \mathcal{L}(y)$, then $\mathcal{L}(y') = sY(s) - y(0)\,,\ \mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)\,.$

18. Convolution. Let $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$. Then $\mathcal{L}(f(t) * g(t)) = F(s)G(s)$