2010 final exam solutions

1a. The direction of most rapid increase is given by the gradient $\nabla f = \langle f_x, f_y, f_z \rangle$, where

$$\begin{aligned} f_x &= \frac{(y^2 - \sin(x+2z))_x}{2\sqrt{y^2 - \sin(x+2z)}} \\ &= \frac{-\cos(x+2z)}{2\sqrt{y^2 - \sin(x+2z)}} = -\frac{\cos 0}{2\sqrt{4 - \sin 0}} = -\frac{1}{4} \end{aligned}$$

at the given point, while a similar computation gives $f_y = 1$ and $f_z = -1/2$. To find a unit vector $\boldsymbol{u}$ in the direction of the gradient, we simply divide by its length:

$$\begin{aligned} \nabla f = \left\langle -\frac{1}{4}, 1, -\frac{1}{2} \right\rangle \quad &\Longrightarrow \quad ||\nabla f|| = \sqrt{\frac{1}{16} + 1 + \frac{1}{4}} = \frac{\sqrt{21}}{4} \\ &\Longrightarrow \quad \boldsymbol{u} = \frac{\nabla f}{||\nabla f||} = \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right\rangle . \end{aligned}$$

1b. Since $\boldsymbol{u}$ is a unit vector in the direction of most rapid increase, $-\boldsymbol{u}$ is a unit vector in the direction of most rapid decrease.

1c. The rate of change in the direction of $\pm\boldsymbol{u}$ is equal to $\pm||\nabla f|| = \pm\frac{1}{4}\sqrt{21}$, respectively.

2a. Let us first simplify the given equation and write

$$z = f(x,y) = \frac{1}{2}\ln(2x^2 + y^2) - \ln 3.$$

Differentiating with respect to x, we then find that

$$f_x = \frac{1}{2} \cdot \frac{4x}{2x^2 + y^2} = \frac{1}{2} \cdot \frac{8}{9} = \frac{4}{9}$$

at the given point, while a similar computation gives $f_y = -1/9$. Noting that

$$z_0 = f(2,-1) = \frac{1}{2}\ln(8+1) - \ln 3 = \frac{1}{2}\ln 3^2 - \ln 3 = 0,$$

we conclude that the equation of the tangent plane at the given point is

$$z = f_x(2,-1) \cdot (x-2) + f_y(2,-1) \cdot (y+1) = \frac{4x - y - 9}{9}.$$

2b. Write the equation of the tangent plane in the form

$$9z = 4x - y - 9 \quad \Longrightarrow \quad 4x - y - 9z = 9.$$

The normal line passes through $(2,-1,0)$ with direction $\langle 4,-1,-9 \rangle$, so its equation is

$$x = 2 + 4t, \qquad y = -1 - t, \qquad z = -9t.$$

3a. The surface lies above the rectangle R, so its projection onto the xy-plane is just R.

3b. Solving the given equation for z, one finds that

$$z^2 = 8 - y^2 \quad \Longrightarrow \quad z = \sqrt{8 - y^2}.$$

In particular, we have $z = f(x, y)$ for some function f and we need to integrate

$$dS = \sqrt{1 + f_x^2 + f_y^2}\; dx\, dy$$

over the region R. Since $f(x, y) = \sqrt{8 - y^2}$ by above, we get

$$f_y = -\frac{2y}{2\sqrt{8 - y^2}} \quad \Longrightarrow \quad 1 + f_x^2 + f_y^2 = 1 + \frac{y^2}{8 - y^2} = \frac{8}{8 - y^2}$$

and so the area of the surface is

$$\text{Area} = \iint_R dS = \int_{-2}^{2} \int_{-1}^{1} \frac{\sqrt{8}\, dx\, dy}{\sqrt{8 - y^2}} = \int_{-2}^{2} \frac{2\sqrt{8}\, dy}{\sqrt{8 - y^2}}.$$

To compute the rightmost integral, we use an analogue of polar coordinates:

$$\begin{aligned} 8 - y^2 = u^2 \quad &\Longrightarrow \quad y^2 + u^2 = 8 \\ &\Longrightarrow \quad y = \sqrt{8} \sin\theta, \qquad u = \sqrt{8} \cos\theta. \end{aligned}$$

Since $y = -2$ when $\sin\theta = -1/\sqrt{2}$ and $y = 2$ when $\sin\theta = 1/\sqrt{2}$, we get

$$\text{Area} = \int_{-2}^{2} \frac{2\sqrt{8}\, dy}{\sqrt{8 - y^2}} = \int_{-\pi/4}^{\pi/4} \frac{2\sqrt{8} \cdot \sqrt{8} \cos\theta\, d\theta}{\sqrt{8} \cos\theta} = \int_{-\pi/4}^{\pi/4} 2\sqrt{8}\, d\theta = 2\pi\sqrt{2}.$$

4a. The given line integral is path-independent because

$$-(2x + 5y + 3)_x = -2 = (3x - 2y + 4)_y.$$

4b. A potential function ϕ is a function that satisfies the equations

$$\phi_x = 3x - 2y + 4, \qquad \phi_y = -2x - 5y - 3.$$

Integrating the first equation with respect to x gives

$$\phi = \int (3x - 2y + 4)\, dx = \frac{3x^2}{2} - 2xy + 4x + C_1(y),$$

and we may similarly integrate the second equation to get

$$\phi = -\int (2x + 5y + 3)\, dy = -2xy - \frac{5y^2}{2} - 3y + C_2(x).$$

Thus, one can always take $\phi(x, y) = \frac{3}{2}x^2 + 4x - \frac{5}{2}y^2 - 3y - 2xy$.

4c. According to the fundamental theorem, the value of the integral is

$$\phi(1,0) - \phi(-1,4) = 11/2 - (-93/2) = 52.$$

5a. Spherical coordinates are defined by the equations

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

5b. The volume of a ball of radius R is given by the triple integral

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_0^R \sin\phi \, d\phi \, d\theta \\ &= \frac{R^3}{3} \int_0^{2\pi} \Big[-\cos\phi \Big]_0^{\pi} \, d\theta = \frac{4\pi R^3}{3}. \end{aligned}$$

To find the mass of the solid between the two spheres, we proceed similarly to get

$$\begin{aligned} \text{Mass} &= \int_0^{2\pi} \int_0^{\pi} \int_2^{3} \rho e^{-\rho^2} \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \left[\frac{e^{-\rho^2}}{-2} \right]_2^3 \sin\phi \, d\phi \, d\theta \\ &= \frac{e^{-4} - e^{-9}}{2} \int_0^{2\pi} \Big[-\cos\phi \Big]_0^{\pi} \, d\theta = 2\pi(e^{-4} - e^{-9}). \end{aligned}$$

6a. Applying the Laplace transform to both sides gives

$$s^2 \mathscr{L}(y) - sy(0) - y'(0) + 4\mathscr{L}(y) = \frac{4e^{-\pi s}}{s} - 4e^{-3\pi s}$$

and we can rearrange terms to write this equation as

$$(s^2+4)\mathscr{L}(y) = s + \frac{4e^{-\pi s}}{s} - 4e^{-3\pi s}.$$

Next, we divide by s^2+4 and use partial fractions to find that

$$\begin{aligned} \mathscr{L}(y) &= \frac{s}{s^2+4} + \frac{4e^{-\pi s}}{s(s^2+4)} - \frac{4e^{-3\pi s}}{s^2+4} \\ &= \frac{s}{s^2+4} + \frac{e^{-\pi s}}{s} - \frac{se^{-\pi s}}{s^2+4} - \frac{4e^{-3\pi s}}{s^2+4}. \end{aligned}$$

Using this fact and our table of Laplace transforms, we conclude that

$$\begin{aligned} y(t) &= \cos(2t) + u(t-\pi) - u(t-\pi)\cos(2t-2\pi) - 2u(t-3\pi)\sin(2t-6\pi) \\ &= \cos(2t) + u(t-\pi) - u(t-\pi)\cos(2t) - 2u(t-3\pi)\sin(2t). \end{aligned}$$

6b. The input function refers to the right hand side of the given equation. In this case, it is 0 when $t < \pi$, it is 4 when $\pi \leq t \neq 3\pi$ and it is minus infinity when $t = 3\pi$. As for the solution we found in part (a), this can be written in the form

$$y(t) = \left\{ \begin{array}{cl} \cos(2t) & \text{if } t < \pi \\ 1 & \text{if } \pi \leq t < 3\pi \\ 1 - 2\sin(2t) & \text{if } t \geq 3\pi \end{array} \right\}.$$

A sketch of the graph of this function appears in the figure below.