

# UNIVERSITY OF DUBLIN

XMA2E011

## TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS  
AND SCIENCE

SCHOOL OF MATHEMATICS

SF Engineers  
SF MSISS  
SF MEMS

Trinity Term 2010

MODULE 2E011

Tuesday, April 27

LUCE LOWER

9.30 — 11.30

Dr. Sergey Frolov

ATTEMPT FIVE QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Consider the function

$$f(x, y, z) = \sqrt{y^2 - \sin(x + 2z)}, \quad \text{and the point } P(0, 2, 0).$$

- (a) 12 marks. Find a unit vector in the direction in which  $f$  increases most rapidly at the point  $P$ .
- (b) 3 marks. Find a unit vector in the direction in which  $f$  decreases most rapidly at the point  $P$ .
- (c) 3 marks. Find the rate of change of  $f$  at the point  $P$  in these directions.

Show the details of your work.

2. Consider the surface

$$z = \ln \frac{\sqrt{2x^2 + y^2}}{3}$$

- (a) 12 marks. Find an equation for the tangent plane to the surface at the point  $P(2, -1, 0)$ .
- (b) 6 marks. Find parametric equations for the normal line to the surface at the point  $P(2, -1, 0)$ .

Show the details of your work.

3. Consider the portion of the cylinder  $y^2 + z^2 = 8$  that is above the rectangle

$$R = \{(x, y) : -1 \leq x \leq 1, -2 \leq y \leq 2\}.$$

- (a) 2 marks. Sketch the projection of the portion onto the  $xy$ -plane.
- (b) 16 marks. Use double integration to find the area of the portion.

Show the details of your work.

4. (a) 3 marks. Show that the integral below is independent of the path

$$\int_{(-1,4)}^{(1,0)} (3x - 2y + 4) dx - (2x + 5y + 3) dy.$$

- (b) 12 marks. Find the potential function  $\phi(x, y)$
- (c) 3 marks. Use the Fundamental Theorem of Line Integrals to find the value of the integral.

Show the details of your work.

5. (a) 5 marks. Express rectangular coordinates in terms of spherical coordinates

- (b) Use triple integral and spherical coordinates to

- i. 6 marks. Compute the volume of a ball of radius  $R$ .

- ii. 7 marks. Find the mass of the solid enclosed between the spheres

$$x^2 + y^2 + z^2 = 4 \text{ and } x^2 + y^2 + z^2 = 9 \text{ if the density is}$$

$$\delta(x, y, z) = \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2 + y^2 + z^2}}.$$

Show the details of your work.

6. (a) 14 marks. Solve the following initial value problem by the Laplace transform

$$y'' + 4y = 4u(t - \pi) - 4\delta(t - 3\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

- (b) 4 marks. Sketch the input function and the solution.

Show the details of your work.

## Useful Formulae

1. Let  $\mathbf{r}(t)$  be a vector function with values in  $\mathbf{R}^3$ :  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ .
  - (a) Its derivative is  $\frac{d\mathbf{r}}{dt} = \left(\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right)$ .
  - (b) The magnitude of this vector is  $\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}$ .
  - (c) The unit tangent vector is  $\mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{\left\|\frac{d\mathbf{r}}{dt}\right\|}$ .
  - (d) The vector equation of the line tangent to the graph of  $\mathbf{r}(t)$  at the point  $P = (x_0, y_0, z_0)$  corresponding to  $t = t_0$  on the curve is  $\mathbf{R}(t) = \mathbf{r}_0 + (t - t_0)\mathbf{v}_0$ , where  $\mathbf{r}_0 = \mathbf{r}(t_0)$  and  $\mathbf{v}_0 = \frac{d\mathbf{r}}{dt}(t_0)$ .
  - (e) The arc length of the graph of  $\mathbf{r}(t)$  between  $t_1$  and  $t_2$  is  $L = \int_{t_1}^{t_2} \left\|\frac{d\mathbf{r}}{dt}\right\| dt$ .
  - (f) The arc length parameter  $s$  having  $\mathbf{r}(t_0)$  as its reference point is  $s = \int_{t_0}^t \left\|\frac{d\mathbf{r}}{du}\right\| du$ .
2. Let  $\sigma$  be a surface in  $\mathbf{R}^3$ :  $z = f(x, y)$ 
  - (a) The slope  $k_x$  of the surface in the  $x$ -direction at the point  $(x_0, y_0)$  is  $k_x = \frac{\partial z}{\partial x}(x_0, y_0)$ .
  - (b) The slope  $k_y$  of the surface in the  $y$ -direction at the point  $(x_0, y_0)$  is  $k_y = \frac{\partial z}{\partial y}(x_0, y_0)$ .
  - (c) The equation for the tangent plane to the surface at the point  $P = (x_0, y_0, z_0)$  is  $z = z_0 + k_x(x - x_0) + k_y(y - y_0)$ .
  - (d) Parametric equations for the normal line to the surface at  $P = (x_0, y_0, z_0)$  are  $\mathbf{r}(t) = \mathbf{r}_0 + t(-k_x\mathbf{i} - k_y\mathbf{j} + \mathbf{k})$ ,  $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ .
  - (e) The volume under the surface and over a region  $R$  in the  $xy$ -plane is  $V = \iint_R f(x, y) dA$ .
  - (f) The area of the portion of the surface that is above a region  $R$  in the  $xy$ -plane is  $S = \iint_\sigma dS = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$ .
  - (g) The mass of the lamina with the density  $\delta(x, y, z)$  that is the portion of the surface that is above a region  $R$  in the  $xy$ -plane is  $M = \iint_\sigma \delta(x, y, z) dS = \iint_R \delta(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$ .

3. The local linear approximation of the function  $z = f(x, y)$  at the point  $(x_0, y_0)$  is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

4. Let  $f(x, y, z)$  be a function of three variables

(a) The gradient of  $f$  is  $\nabla f = (f_x, f_y, f_z)$ .

(b)  $f$  increases most rapidly in the direction of its gradient, and the rate of change of  $f$  in this direction is equal to  $\|\nabla f\|$ .

(c) If  $f$  is smooth then its critical points satisfy  $f_x = f_y = f_z = 0$ .

5. Let  $R$  be a region in the  $xy$ -plane bounded by the curves  $y = g(x)$ ,  $y = h(x)$ ,  $x = a$ ,  $x = b$ , and  $g \leq h$  for  $a \leq x \leq b$ . Then the double integral over the region is

$$\iint_R f(x, y) dA = \int_a^b \left[ \int_{g(x)}^{h(x)} f(x, y) dy \right] dx.$$

6. Let  $R$  be a region in the  $xy$ -plane bounded by the curves (in polar coordinates)

$r = r_1(\theta)$ ,  $r = r_2(\theta)$ ,  $\theta = \alpha$ ,  $\theta = \beta$  and  $r_1 \leq r_2$  for  $\alpha \leq \theta \leq \beta$ . Then the double integral over the region is

$$\iint_R f(r, \theta) dA = \iint_R f(r, \theta) r dr d\theta = \int_\alpha^\beta \left[ \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r dr \right] d\theta.$$

7. Let  $R$  be a plain lamina with density  $\delta(x, y)$ .

(a) Its mass is equal to  $M = \iint_R \delta(x, y) dA$ .

(b) The  $x$ -coordinate of its centre of gravity is equal to  $x_{cg} = \frac{1}{M} \iint_R x \delta(x, y) dA$ .

(c) The  $y$ -coordinate of its centre of gravity is equal to  $y_{cg} = \frac{1}{M} \iint_R y \delta(x, y) dA$ .

8. Let  $G$  be a simple solid whose projection onto the  $xy$ -plane is a region  $R$ .  $G$  is bounded by a surface  $z = g(x, y)$  from below and by a surface  $z = h(x, y)$  from above.

(a) The triple integral over the solid is  $\iiint_G f(x, y, z) dV = \iint_R \left[ \int_{g(x, y)}^{h(x, y)} f(x, y, z) dz \right] dA$ .

(b) The volume of the solid is  $V = \iiint_G dV = \iint_R [h(x, y) - g(x, y)] dA$ .

9. Let  $G$  be a solid enclosed between the two surfaces (in spherical coordinates)

$$r = g(\theta, \phi), \quad r = h(\theta, \phi).$$

- (a) The triple integral over the solid is

$$\iiint_G f(r, \theta, \phi) dV = \int_0^{2\pi} \left( \int_0^\pi \left[ \int_{g(\theta, \phi)}^{h(\theta, \phi)} f(r, \theta, \phi) r^2 dr \right] \sin \phi d\phi \right) d\theta.$$

- (b) The volume of the solid is  $V = \iiint_G dV = \int_0^{2\pi} \left( \int_0^\pi \left[ \int_{g(\theta, \phi)}^{h(\theta, \phi)} r^2 dr \right] \sin \phi d\phi \right) d\theta.$

- (c) The mass of the solid with the density  $\delta(r, \theta, \phi)$  is  $M = \iiint_G \delta(r, \theta, \phi) dV.$

10. Let a region  $R_{xy}$  in the  $xy$ -plane be mapped to a region  $R_{uv}$  in the  $uv$ -plane under the change of variables  $u = u(x, y)$ ,  $v = v(x, y)$ .

- (a) The magnitude of the Jacobian of the change is  $\left| \frac{\partial(u, v)}{\partial(x, y)} \right| = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right|.$

- (b) The integral over  $R_{xy}$  is  $\iint_{R_{xy}} f(x, y) dA_{xy} = \iint_{R_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1} dA_{uv}.$

11. The area of the surface that extends upward from the curve  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$  in the  $xy$ -plane to the surface  $z = f(x, y)$  is given by the following line integral

$$A = \int_C z ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

12. Consider a line integral  $\int_C f(x, y) dx + g(x, y) dy$ , and let  $P = (x_P, y_P)$  and  $Q = (x_Q, y_Q)$  be the endpoints of the curve  $C$ .

- (a) The line integral is independent of the path if  $\partial_y f(x, y) = \partial_x g(x, y).$

- (b) Then there is a potential function  $\phi(x, y)$  satisfying  $\frac{\partial \phi}{\partial x} = f(x, y)$ ,  $\frac{\partial \phi}{\partial y} = g(x, y),$

- (c) and the Fundamental Theorem of Line Integrals says that

$$\int_C f(x, y) dx + g(x, y) dy = \int_P^Q \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \phi(x, y)|_P^Q = \phi(x_Q, y_Q) - \phi(x_P, y_P).$$

13. Let a closed curve  $C$  be oriented counterclockwise, and be the boundary of a simply connected region  $R$  in the  $xy$ -plane. By Green's Theorem we have

$$\oint_C f(x, y) dx + g(x, y) dy = \iint_R \left( \frac{\partial g(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y} \right) dA$$

14. Let  $\mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$  be a vector field.

- (a) If  $\sigma$  is the surface  $z = f(x, y)$ , oriented by upward unit normals  $\mathbf{n}$ , and  $R$  is the projection of  $\sigma$  onto the  $xy$ -plane then

$$\text{flux} = \iint_\sigma \mathbf{F} \cdot \mathbf{n} dS = \iint_R \left( -M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \right) dA.$$

- (b) If  $\sigma$  is the surface  $z = f(x, y)$ , oriented by downward unit normals  $\mathbf{n}$ , and  $R$  is the projection of  $\sigma$  onto the  $xy$ -plane then

$$\text{flux} = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \left( M \frac{\partial f}{\partial x} + N \frac{\partial f}{\partial y} - P \right) dA.$$

- (c) According to the Divergence Theorem the flux of  $\mathbf{F}$  across a closed surface  $\sigma$  with outward orientation is

$$\text{flux} = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \text{div } \mathbf{F} dV, \quad \text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

- (d) If  $\sigma$  is an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve  $C$  with positive orientation then, according to Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS, \quad \text{curl } \mathbf{F} = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

15. The Laplace transform of a function  $f(t)$  is the function  $F(s)$  defined by

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt, \quad f(t) = \mathcal{L}^{-1}(F(s)).$$

Function	Transform	Function	Transform
$e^{at}$	$\frac{1}{s-a}$	$e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at} \sinh \omega t$	$\frac{\omega}{(s-a)^2 - \omega^2}$	$e^{at} \cosh \omega t$	$\frac{s-a}{(s-a)^2 - \omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$	$\delta(t-a)$	$e^{-as}$

16. Let  $F(s) = \mathcal{L}(f(t))$ , then  $\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$ ;

$$\mathcal{L}(e^{at}f(t)) = F(s-a); \quad \mathcal{L}(tf(t)) = -\frac{dF(s)}{ds}; \quad \mathcal{L}(f(kt)) = \frac{1}{k}F\left(\frac{s}{k}\right).$$

17. Let  $Y(s) = \mathcal{L}(y)$ , then  $\mathcal{L}(y') = sY(s) - y(0)$ ,  $\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$ .

18. Convolution. Let  $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau) d\tau$ . Then  $\mathcal{L}(f(t) * g(t)) = F(s)G(s)$