Lecture 29, December 10

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s)$	t^n	$n!/s^{n+1}$
e^{kt}	$1/(s-k)$	$\sin(kt)$	$k/(s^2+k^2)$
$e^{kt}f(t)$	$F(s-k)$	$u(t-k)f(t-k)$	$e^{-ks}F(s)$
$\delta(t-k)$	e^{-ks}	$u(t-k)$	e^{-ks}/s

. . . .

Example 1. We use the table above to solve the initial value problem

$$y''(t) + y(t) = \delta(t-1) + u(t-2), \qquad y(0) = 0, \qquad y'(0) = 1.$$

Taking the Laplace transform of both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + \mathscr{L}(y) = e^{-s} + \frac{e^{-2s}}{s}$$

and we can solve for $\mathscr{L}(y)$ to find that

$$(s^2+1)\mathscr{L}(y) = 1 + e^{-s} + \frac{e^{-2s}}{s} \quad\Longrightarrow\quad \mathscr{L}(y) = \frac{1}{s^2+1} + \frac{e^{-s}}{s^2+1} + \frac{e^{-2s}}{s(s^2+1)}.$$

To handle the rightmost term, we have to decompose it into partial fractions as

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Let us now determine the coefficients A, B, C. Clearing denominators gives

$$1 = A(s^2+1) + (Bs+C)s = As^2 + A + Bs^2 + Cs$$

and we may compare coefficients of s to find that

$$A = 1, \qquad C = 0, \qquad B = -A = -1.$$

This gives rise to the partial fractions decomposition

$$\mathscr{L}(y) = \frac{1}{s^2+1} + \frac{e^{-s}}{s^2+1} + \frac{e^{-2s}}{s} - \frac{se^{-2s}}{s^2+1}.$$

Consulting the table once again, we conclude that

$$y(t) = \sin t + u(t-1)\sin(t-1) + u(t-2) - u(t-2)\cos(t-2).$$