Lecture 28, December 7

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s)$	t^n	$n!/s^{n+1}$
e^{kt}	$1/(s-k)$	$\sin(kt)$	$k/(s^2+k^2)$
$e^{kt}f(t)$	$F(s-k)$	$u(t-k)f(t-k)$	$e^{-ks}F(s)$

. . . .

Example 1. We compute $\mathscr{L}(t^2e^{4t})$. Ignoring the exponential factor, we find that

$$\mathscr{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}.$$

If we now include the exponential factor e^{4t}, then s becomes $s-4$ and we get

$$\mathscr{L}(t^2e^{4t}) = \frac{2}{(s-4)^3}.$$

Example 2. We compute $\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s^2-5s+6}\right)$. First, we use partial fractions to write

$$\frac{1}{s^2-5s+6} = \frac{1}{(s-3)(s-2)} = \frac{1}{s-3} - \frac{1}{s-2}$$

and we consult our table to find that

$$\mathscr{L}^{-1}\left(\frac{1}{s^2-5s+6}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-3}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}.$$

If we now include the exponential factor e^{-2s}, then t becomes $t-2$ and we get

$$\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s^2-5s+6}\right) = u(t-2)\cdot(e^{3t-6} - e^{2t-4}).$$

Example 3. We compute $\mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}\right)$. Since the denominator does not factor, we cannot use partial fractions in this case. Let us then complete the square to express

$$\frac{1}{s^2-4s+5} = \frac{1}{s^2-4s+4+1} = \frac{1}{(s-2)^2+1}$$

as a shifted version of $\frac{1}{s^2+1}$. According to our table, this implies

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \quad\Longrightarrow\quad \mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}\right) = e^{2t}\sin t.$$