Lecture 27, December 5

- **Laplace transform.** Some of its main properties are listed in the following table.

Function	Laplace transform	Function	Laplace transform
1	$1/s$	e^{kt}	$1/(s-k)$
$y'(t)$	$s\mathscr{L}(y) - y(0)$	$\sin(kt)$	$k/(s^2+k^2)$
$y''(t)$	$s^2\mathscr{L}(y) - sy(0) - y'(0)$	$\cos(kt)$	$s/(s^2+k^2)$

. . . .

Example 1. We use the table above to solve the initial value problem

$$y''(t) + 4y(t) = 2e^t, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Taking the Laplace transform of both sides gives

$$s^2\mathscr{L}(y) - sy(0) - y'(0) + 4\mathscr{L}(y) = \frac{2}{s-1}$$

and we can solve for $\mathscr{L}(y)$ to find that

$$(s^2+4)\mathscr{L}(y) = s + \frac{2}{s-1} \quad\implies\quad \mathscr{L}(y) = \frac{s}{s^2+4} + \frac{2}{(s-1)(s^2+4)}.$$

To handle the rightmost term, we have to decompose it into partial fractions as

$$\frac{2}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}.$$

Let us now determine the coefficients A, B, C. Clearing denominators gives

$$2 = A(s^2+4) + (Bs+C)(s-1)$$

and this identity should hold for all s. When $s = 1$, the identity reduces to

$$2 = 5A \quad\implies\quad A = 2/5.$$

When $s = 0$, we get $2 = 4A - C$ and so $C = 4A - 2 = -2/5$. When $s = -1$, we get

$$2 = 5A - 2(C - B) = 2 + 4/5 + 2B \quad\implies\quad B = 1 - 1 - 2/5 = -2/5.$$

We now employ this partial fractions decomposition to write

$$\mathscr{L}(y) = \frac{s}{s^2+4} + \frac{2/5}{s-1} - \frac{2s/5}{s^2+4} - \frac{2/5}{s^2+4}.$$

Consulting the table once again, we conclude that

$$\begin{aligned} y(t) &= \cos(2t) + \frac{2e^t}{5} - \frac{2\cos(2t)}{5} - \frac{\sin(2t)}{5} \\ &= \frac{3\cos(2t)}{5} + \frac{2e^t}{5} - \frac{\sin(2t)}{5}. \end{aligned}$$