Lecture 26, December 3

- **Laplace transform.** Given a function $f(t)$, we define $\mathscr{L}(f)$ by the formula

$$\mathscr{L}(f) = \int_0^\infty e^{-st} f(t)\, dt, \qquad s > 0.$$

Some of the properties of the Laplace transform are listed in the following table.

Function	Laplace transform
1	$1/s$
e^{kt}	$1/(s-k)$
$f'(t)$	$s\mathscr{L}(f) - f(0)$

In addition, $\mathscr{L}(f+g) = \mathscr{L}(f) + \mathscr{L}(g)$ and $\mathscr{L}(cf) = c\mathscr{L}(f)$ for each constant c.

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Example 1. The Laplace transform of the function $y(t) = 2e^t + 3e^{2t}$ is given by

$$\mathscr{L}(y) = 2\mathscr{L}(e^t) + 3\mathscr{L}(e^{2t}) = \frac{2}{s-1} + \frac{3}{s-2} = \frac{5s-7}{(s-1)(s-2)}.$$

Example 2. We use the Laplace transform to solve $y'(t) = y(t)$ subject to $y(0) = y_0$. Taking the Laplace transform of both sides, we find that

$$\mathscr{L}(y') = \mathscr{L}(y) \quad \Longrightarrow \quad s\mathscr{L}(y) - y_0 = \mathscr{L}(y).$$

Next, we solve for $\mathscr{L}(y)$ and consult the table to conclude that

$$(s-1)\mathscr{L}(y) = y_0 \quad \Longrightarrow \quad \mathscr{L}(y) = \frac{y_0}{s-1} \quad \Longrightarrow \quad y(t) = y_0 e^t.$$

Example 3. We use the Laplace transform to solve $y'(t) - 2y(t) = 4$ subject to the initial condition $y(0) = 1$. Taking the Laplace transform of both sides gives

$$\begin{aligned} \mathscr{L}(y') - 2\mathscr{L}(y) = \mathscr{L}(4) \quad &\Longrightarrow \quad s\mathscr{L}(y) - y(0) - 2\mathscr{L}(y) = \frac{4}{s} \\ &\Longrightarrow \quad (s-2)\mathscr{L}(y) = y(0) + \frac{4}{s} = 1 + \frac{4}{s}. \end{aligned}$$

To use the table in this case, one needs to employ partial fractions to write

$$\mathscr{L}(y) = \frac{1}{s-2} + \frac{4}{s(s-2)} = \frac{1}{s-2} + \frac{2}{s-2} - \frac{2}{s} = \frac{3}{s-2} - \frac{2}{s}$$

and this is easily seen to imply that $y(t) = 3e^{2t} - 2$.