Lecture 25, November 30

- **Stokes' theorem.** If σ is an oriented surface that is bounded by the curve C and C is positively oriented (according to the right hand rule), then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS.$$

And if σ is the graph of $z = f(x,y)$ oriented upwards, then $\boldsymbol{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy$.

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Example 1. Let $\boldsymbol{F} = \langle z, x, y\rangle$ and let σ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upwards. In this case, we have

$$z = f(x,y) = 1 - x^2 - y^2 \quad \Longrightarrow \quad \boldsymbol{n}\, dS = \langle 2x, 2y, 1\rangle\, dx\, dy$$

and one can easily check that

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} = \langle 1,1,1\rangle .$$

Taking the dot product of these two vectors, we conclude that

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \iint_\sigma (2x + 2y + 1)\, dx\, dy = \int_0^{2\pi}\int_0^1 (2r^2\cos\theta + 2r^2\sin\theta + r)\, dr\, d\theta \\ &= \int_0^{2\pi}\left(\frac{2\cos\theta}{3} + \frac{2\sin\theta}{3} + \frac{1}{2}\right) d\theta = \left[\frac{2\sin\theta}{3} - \frac{2\cos\theta}{3} + \frac{\theta}{2}\right]_0^{2\pi} = \pi.\end{aligned}$$

Example 2. Let $\boldsymbol{F} = \langle z, x, y\rangle$ and let σ be the part of the plane $x + 2y + z = 4$ that lies in the first octant, oriented upwards. Arguing as before, we get

$$z = f(x,y) = 4 - x - 2y \quad \Longrightarrow \quad \boldsymbol{n}\, dS = \langle 1, 2, 1\rangle\, dx\, dy$$

as well as $\operatorname{curl} \boldsymbol{F} = \langle 1,1,1\rangle$, so Stokes' theorem implies that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_\sigma (1 + 2 + 1)\, dx\, dy = 4 \iint_\sigma dx\, dy.$$

The values of x, y are determined by the projection onto the xy-plane. This is formed by the line $x + 2y = 4$ (that we get when $z = 0$) and the coordinate axes, hence

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 4\int_0^2 \int_0^{4-2y} dx\, dy &= 4\int_0^2 (4 - 2y)\, dy \\ &= 4\Big[4y - y^2\Big]_0^2 = 16.\end{aligned}$$