Lecture 24, November 28

- **Divergence theorem.** The outward flux of $\boldsymbol{F}$ through a closed surface σ in $\mathbb{R}^3$ is

$$\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_G (\operatorname{div}\boldsymbol{F})\,dV,$$

where $\boldsymbol{n}$ is the outward unit normal vector and G is the solid enclosed by σ.

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Example 1. Let $\boldsymbol{F} = \langle 2x, 3y, z^2\rangle$ and let σ be the surface consisting of the six faces of the unit cube. Then the outward flux of $\boldsymbol{F}$ through σ is given by

$$\begin{aligned}\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS &= \iiint_G (2+3+2z)\,dV = \int_0^1\int_0^1\int_0^1 (5+2z)\,dx\,dy\,dz\\ &= \int_0^1\int_0^1 (5+2z)\,dy\,dz = \int_0^1 (5+2z)\,dz = 5+1 = 6.\end{aligned}$$

Example 2. Let $\boldsymbol{F} = \langle x^3, y^3, z^2\rangle$ and let σ be the surface of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=1$ (including the top and bottom parts). Then the outward flux through this surface may be computed as

$$\begin{aligned}\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS &= \iiint_G (3x^2+3y^2+2z)\,dV = \int_0^1\int_0^{2\pi}\int_0^2 (3r^3+2rz)\,dr\,d\theta\,dz\\ &= \int_0^1\int_0^{2\pi}\left[\frac{3r^4}{4}+r^2z\right]_{r=0}^2 d\theta\,dz = \int_0^1\int_0^{2\pi}(12+4z)\,d\theta\,dz\\ &= \int_0^1 2\pi(12+4z)\,dz = 2\pi(12+2) = 28\pi.\end{aligned}$$

Example 3. Let $\boldsymbol{F} = \langle y, x, z\rangle$ and let σ be the sphere $x^2+y^2+z^2=a^2$ of radius a around the origin. Then the outward flux through σ is given by

$$\iint_\sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_G (0+0+1)\,dV = \iiint_G dV = \text{volume of } G = \frac{4\pi a^3}{3}.$$