Lecture 23, November 26

- **Flux.** The flux of the vector field $\boldsymbol{F}(x, y, z)$ through a surface σ in $\mathbb{R}^3$ is

$$\text{Flux} = \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n} \, dS,$$

where $\boldsymbol{n}$ is the unit normal vector depending on the orientation of the surface. If σ is the graph of $z = f(x, y)$ oriented upwards, then $\boldsymbol{n} \, dS = \langle -f_x, -f_y, 1 \rangle \, dx \, dy$.

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Example 1. Let $\boldsymbol{F} = \langle x, y, z \rangle$ and let σ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upwards. In this case, we have

$$f(x, y) = 1 - x^2 - y^2 \quad \Longrightarrow \quad \boldsymbol{n} \, dS = \langle 2x, 2y, 1 \rangle \; dx \, dy.$$

Taking the dot product with $\boldsymbol{F} = \langle x, y, 1 - x^2 - y^2 \rangle$, we end up with

$$\begin{aligned} \text{Flux} &= \iint (2x^2 + 2y^2 + 1 - x^2 - y^2) \, dx \, dy \\ &= \iint (x^2 + y^2 + 1) \, dx \, dy \end{aligned}$$

and the projection of σ onto the xy-plane is the interior of the circle $x^2 + y^2 = 1$, so

$$\begin{aligned} \text{Flux} &= \int_0^{2\pi} \int_0^1 (r^2 + 1) \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r^3 + r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^2}{2} \right]_{r=0}^1 d\theta = \int_0^{2\pi} \frac{3}{4} \, d\theta = \frac{3\pi}{2}. \end{aligned}$$

Example 2. Let $\boldsymbol{F} = \langle 1, y, 0 \rangle$ and let σ be the part of the plane $x + y + z = 1$ that lies in the first octant, oriented upwards. Then $z = f(x, y) = 1 - x - y$ and

$$\boldsymbol{n} \, dS = \langle -f_x, -f_y, 1 \rangle \; dx \, dy = \langle 1, 1, 1 \rangle \; dx \, dy.$$

Taking the dot product with $\boldsymbol{F} = \langle 1, y, 0 \rangle$, we conclude that

$$\begin{aligned} \text{Flux} &= \iint (1 + y) \, dx \, dy = \int_0^1 \int_0^{1-y} (1 + y) \, dx \, dy \\ &= \int_0^1 (1 + y)(1 - y) \, dy = \int_0^1 (1 - y^2) \, dy = 1 - \frac{1}{3} = \frac{2}{3}. \end{aligned}$$