Lecture 22, November 23

- **Surface integrals.** The integral of $f(x, y, z)$ over a surface σ in $\mathbb{R}^3$ is

$$\iint_\sigma f(x, y, z)\, dS = \iint f(x(u, v), y(u, v), z(u, v)) \cdot ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, du\, dv,$$

where $\boldsymbol{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is the parametric equation of the surface.

- When the surface is the graph of $z = f(x, y)$, one has $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$.

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Example 1 (Cylinder). The parametric equation of the cylinder $x^2 + y^2 = 1$ is

$$\boldsymbol{r} = \langle x, y, z\rangle = \langle \cos\theta, \sin\theta, z\rangle$$

and it is obtained using cylindrical coordinates. In this case, we have

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_z &= \langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0, 0, 1\rangle = \langle \cos\theta, \sin\theta, 0\rangle\,, \\ ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| &= \sqrt{\cos^2\theta + \sin^2\theta} = 1 \end{aligned}$$

and one can use these facts to compute any surface integral over the cylinder.

Example 2 (Cone). The parametric equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$x^2 + y^2 = z^2 \quad \Longrightarrow \quad \boldsymbol{r} = \langle x, y, z\rangle = \langle z\cos\theta, z\sin\theta, z\rangle\,.$$

To compute a surface integral over the cone, one needs to compute

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_z &= \langle -z\sin\theta, z\cos\theta, 0\rangle \times \langle \cos\theta, \sin\theta, 1\rangle = \langle z\cos\theta, z\sin\theta, -z\rangle\,, \\ ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| &= \sqrt{z^2\cos^2\theta + z^2\sin^2\theta + z^2} = z\sqrt{2}. \end{aligned}$$

Example 3 (Sphere). The parametric equation of the sphere $x^2 + y^2 + z^2 = 1$ is

$$\boldsymbol{r} = \langle x, y, z\rangle = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

and it is obtained using spherical coordinates. In this case, we have

$$\begin{aligned} \boldsymbol{r}_\theta \times \boldsymbol{r}_\phi &= \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle \times \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle \\ &= \left\langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\right\rangle = -(\sin\phi)\boldsymbol{r} \end{aligned}$$

and the fact that $||\boldsymbol{r}|| = 1$ implies that $||\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi|| = \sin\phi$.

Example 4 (A general example). The graph of $z = f(x, y)$ can be described by

$$\boldsymbol{r} = \langle x, y, z \rangle = \langle x, y, f(x, y) \rangle .$$

To compute a surface integral over this graph, one needs to compute

$$\begin{aligned} \boldsymbol{r}_x \times \boldsymbol{r}_y &= \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle , \\ ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| &= \sqrt{1 + f_x^2 + f_y^2}. \end{aligned}$$

Example 5. We compute the integral $\iint_\sigma z^2 \, dS$ in the case that σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between $z = 0$ and $z = 1$. As in Example 2, we have

$$\boldsymbol{r} = \langle z \cos\theta, z \sin\theta, z \rangle , \qquad ||\boldsymbol{r}_\theta \times \boldsymbol{r}_z|| = z\sqrt{2}.$$

This implies $dS = z\sqrt{2} \, dz \, d\theta$, so the given integral becomes

$$\iint_\sigma z^2 \, dS = \int_0^{2\pi} \int_0^1 z^3\sqrt{2} \, dz \, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{4} \, d\theta = \frac{\pi\sqrt{2}}{2}.$$

Example 6. Consider the lamina that occupies the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$. If its density is given by $\delta(x, y, z)$, then its mass is

$$\text{Mass} = \iint_\sigma \delta(x, y, z) \, dS.$$

Assume that δ is constant for simplicity. Since $z = f(x, y) = x^2 + y^2$, we have

$$||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4x^2 + 4y^2}$$

by Example 4. Using this fact and switching to polar coordinates, we find that

$$\begin{aligned} \text{Mass} &= \int\!\!\int \delta\sqrt{1 + 4x^2 + 4y^2} \, dx \, dy \\ &= \delta \int_0^{2\pi} \int_0^1 (1 + 4r^2)^{1/2} \, r \, dr \, d\theta \\ &= \frac{\delta}{8} \int_0^{2\pi} \int_1^5 u^{1/2} \, du \, d\theta \qquad \boxed{u = 1 + 4r^2} \\ &= \frac{\delta}{8} \int_0^{2\pi} \frac{5^{3/2} - 1^{3/2}}{3/2} \, d\theta \\ &= \frac{\delta\pi}{6} (5\sqrt{5} - 1). \end{aligned}$$