Lecture 21, November 21

- **Green's theorem.** If R is a simply connected region in $\mathbb{R}^2$ whose boundary C is a simple, closed piecewise smooth curve oriented counterclockwise, then

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

In particular, the area of the region R may be computed using any of the formulas

$$\text{Area} = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

....

Example 1. Consider the triangle C whose vertices are $(0,0)$, $(1,0)$ and $(1,2)$. Then

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_R (2xy^3 - x)\,dA,$$

where R is the interior of the triangle. This actually gives

$$\oint_C xy\,dx + x^2y^3\,dy = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx = \int_0^1 \left[\frac{2xy^4}{4} - xy\right]_{y=0}^{2x} dx$$

$$= \int_0^1 (8x^5 - 2x^2)\,dx = \frac{8}{6} - \frac{2}{3} = \frac{2}{3}.$$

Example 2. Let C be the circle of radius 2 around the origin and let

$$\boldsymbol{F}(x,y) = \left\langle e^x - y^3, \cos y + x^3 \right\rangle.$$

According to Green's theorem, we then have

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R (3x^2 + 3y^2)\,dA,$$

where R is the interior of the circle. Switching to polar coordinates, we find that

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi} \int_0^2 3r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^4}{4}\right]_{r=0}^2 d\theta = \int_0^{2\pi} 12\,d\theta = 24\pi.$$