Lecture 17, November 2

- **Formula for change of variables.** When it comes to double integrals, one has

$$\iint f(x,y)\,dx\,dy = \iint f(x(u,v),y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Here, the additional factor inside the integral is the absolute value of the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

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Example 1. Consider the region R in the xy-plane bounded by the lines

$$x+y=1, \qquad x+y=2, \qquad x-y=0, \qquad x-y=1.$$

If we introduce the variables $u=x-y$ and $v=x+y$, then we can write

$$\iint_R \frac{x-y}{x+y}\,dx\,dy = \int_1^2 \int_0^1 \frac{u}{v} \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

To compute the Jacobian in this case, we first need to solve for x and y, namely

$$\left\{ \begin{array}{c} u=x-y \\ v=x+y \end{array} \right\} \implies \left\{ \begin{array}{c} u+v=2x \\ v-u=2y \end{array} \right\} \implies \left\{ \begin{array}{c} x=(u+v)/2 \\ y=(v-u)/2 \end{array} \right\}.$$

This allows us to differentiate x,y with respect to u,v and we now get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Keeping this in mind, we can finally compute the given integral as

$$\begin{aligned} \iint_R \frac{x-y}{x+y}\,dx\,dy &= \int_1^2 \int_0^1 \frac{u}{2v}\,du\,dv \\ &= \int_1^2 \left[\frac{u^2}{4v}\right]_{u=0}^1 dv = \int_1^2 \frac{1}{4v}\,dv = \left[\frac{\ln v}{4}\right]_1^2 = \frac{\ln 2}{4}. \end{aligned}$$