Lecture 20, November 16

- **Conservative vector fields.** We say that $\boldsymbol{F} = \langle F_1, F_2 \rangle$ is conservative, if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

In that case, $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$ for some function ϕ (the potential function) and

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla\phi \cdot d\boldsymbol{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

for any curve C from (x_0, y_0) to (x_1, y_1). Thus, the integral is path-independent.

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Example 1. Take $\boldsymbol{F} = \langle 2xy, x^2 + 2y \rangle$. This vector field is conservative because

$$\frac{\partial F_1}{\partial y} = (2xy)_y = 2x, \qquad \frac{\partial F_2}{\partial x} = (x^2 + 2y)_x = 2x.$$

In particular, $\boldsymbol{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$ for some function ϕ and this means that

$$\phi_x = 2xy, \qquad \phi_y = x^2 + 2y.$$

To actually find the potential function ϕ, we note that integration gives

$$\phi = \int 2xy\,dx = x^2 y + C_1(y),$$
$$\phi = \int (x^2 + 2y)\,dy = x^2 y + y^2 + C_2(x)$$

and then compare these two equations to get the potential function $\phi = x^2 y + y^2$.

Example 2. Let $\boldsymbol{F} = \langle 2xy, x^2 + 2y \rangle$ as before and consider the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r},$$

where C is the straight line from $(1, 0)$ to $(0, 1)$. Then we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{(1,0)}^{(0,1)} \nabla\phi \cdot d\boldsymbol{r} = \phi(0, 1) - \phi(1, 0) = 1.$$